A Comparison of the Homotopy Analysis Method and the Homotopy Perturbation Method for the Korteweg-de Vries (K-dV) Equation

Md. Mizanur Rahman, Md. Masum Murshed, Nasima Akhter

Abstract— In this study, the performance of the Homotopy Analysis Method (HAM) and the Homotopy Perturbation Method (HPM) has been compared for the Korteweg-de Vries (K-dV) equation. The solutions of the K-dV equation by HAM and HPM for three cases have been computed using our MATLAB routine. For a suitable exact solution l_2 -errors has been computed for both the methods. The results show that HPM performs better than HAM for small values of time t and both the results agree well with the exact solution for all the three cases.

Keywords— Homotopy Analysis Method (HAM), Homotopy Perturbation Method (HPM), Korteweg-de Vries (K-dV) equation, MATLAB, partial differential equation (PDE), Auxiliary Parameter, Auxiliary Function, Homotopy Parameter.

1 INTRODUCTION

MOST of the natural phenomena are usually expressed by nonlinear partial differential equations (PDEs). The K-dV equation, given by

$$u_{\mathrm{t}} + 6uu_{\mathrm{x}} + u_{\mathrm{xxx}} = 0,$$

where $u(x,t)$ represents unknown function, t represents the time, and the subscripts denote partial differentiation. This equation was first used in [21] to represent low-amplitude water wave in shallow, parochial channels such as canals (see [20]). Many researchers have been used different method to solve different types of K-dV equations for various purposes. Comparison of caputo and conformable derivatives for time-fractional K-dV equation is studied in [6]. In [8], soliton molecules, nonlocal symmetry and CRE method of the K-dV equation is studied. The K-dV equation is studied for water waves in [10, 11]. The structure of unsteady K-dV model arising in shallow water has been studied in [19]. A comparative analysis of the fractional-order coupled K-dV equations with the Mittag–Leffler Law has been studied in [24]. Fractional forced K-dV equation is studied in [29]. A new localized and periodic solution to a K-dV equation with power law nonlinearity has been studied in [32]. N-soliton solutions and dynamic property analysis of a generalized three-component Hirota-Satsuma coupled K-dV equation has been studied in [38]. Moreover, the K-dV equation has now been used to solve variety of problems of different fields including physics, plasma physics and engineering. It is known that most of the nonlinear PDEs do not have analytic solution. For this reason, semi-analytic or numerical solutions are used to solve such problems. There are many semi-analytic methods for the solutions of nonlinear PDEs. Among them HAM and HPM are the most popular. HAM was first introduced by Shijun Liao in [34] considering the ideas of homotopy in the general topology. Furthermore, a group of researchers have successfully employed HAM for variety of nonlinear-problems such as: Generalized Sylvester matrix equation with applications is studied in [2]. In [4] two dimensional linear Volterra fuzzy integral equations have been studied. Non-similar solution of Eyring–Powell fluid flow and heat transfer with convective boundary condition is studied in [5]. Davey-Stewartson equations is studied in [12]. Fuzzy impulsive fractional differential equations have been studied in [25]. In [28] determining the thermal response of convective-radiative porous fins with temperature-dependent properties is studied. SIR epidemic model with Crowley-Martin type functional eesponse And Holling type-Ⅱ treatment rate has been studied in [30]. Beyond Perturbation: Introduction to the Homotopy Analysis Method is studied in [33]. In [35] $\nabla^2 u = b(x,y)$ type equations have been studied. And many other types of nonlinear problems are studied.

The HPM is another popular semi-analytic technique for the solution of nonlinear PDEs. It was introduced by Ji-Huan He in [18] using the general concepts of homotopy in topology. HPM has been utilized by many researchers to solve various types of linear and non-linear problems, such as: Mathematical study of diabetes and its complication is studied in [1]. SIR Mumps_Model has been studied in [3]. In [7] nonlinear Volttra partial integro-differential equations have been studied. Nonlinear Schrodinger equations are studied in [9, 13, 14, 16]. Nonlinear Burger equations are studied in [13, 14]. Nonlinear equations arising in heat transforms is studied in [13]. Fractional differential equations: part 1 Mohand transform is studied in [26]. In [27] nonlinear Oscillators is studied. Biological population model has been studied in [31]. In [37] inverse analysis of Jeffery-Hamel flow problem is studied. And many other types of nonlinear problems have been studied.

- *M. M. Rahman, Department of Mathematics, University of Rajshahi, Rajshahi-6205, Bangladesh. E-mail: mizanur.ru.math@gmail.com*
- *M. M. Murshed, Associate Professor, Department of Mathematics, University of Rajshahi, Rajshahi-6205, Bangladesh. E-mail: mmmurshed82@gmail.com*
- *N. Akhter, Professor, Department of Mathematics, University of Rajshahi, Rajshahi-6205, Bangladesh. Email: nasima.math.ru@gmail.com*

In this study, we have intended to compare the performance of both HAM and HPM for the K-dV equation and compare both the results with a suitable exact solution for different number of terms and different values of parameters. We have considered three different cases. The results show that HPM performs better than HAM for small values of time t, and both the results agreed well with the exact solution for all the three cases.

2 Mathematical background:

2.1 Korteweg-de Vries (K-dV) Equation: In this study we have considered the following K-dV equation,

$$u_t + 6uu_x + u_{xxx} = 0 \qquad \ldots\ \ldots\ \ldots \text{(a)}$$

with the initial approximation $u(x,0) = N(N+1)sech^2(x)$, $N > 0$, where $u(x,t)$ represents unknown function, t represents the time, and the subscripts in equation (a) denote partial differentiation. Considering $N = 1$, $u(x,0) = N(N+1)sech^2(x)$ becomes $u(x,0) = 2sech^2(x)$.

2.2 The Homotopy Analysis Method (HAM):

To demonstrate the fundamental concepts of the HAM, we assume

$$\mathcal{A}[v(t)] - f(t) = 0,\ \ t \in \Omega \qquad \ldots\ \ldots\ \ldots\ (1)$$

be the usual differential equation with boundary condition $\mathcal{B}(v, \frac{\partial v}{\partial m}) = 0$, $t \in \Gamma$, where $\mathcal{A}$ denotes the nonlinear differential operator, $f(t)$ denotes the known function, $\mathcal{B}$ denotes the boundary operator, $v(t)$ represents the unknown function, t is the time, and Γ denotes boundary of the region Ω. The nonlinear differential operator $\mathcal{A}$ can be separated into two parts which are $\mathcal{A}(v) = \mathcal{L}(v) + \mathcal{N}(v)$, where $\mathcal{L}$ represents linear operator, and $\mathcal{N}$ represents non-linear operator. Therefore the equation (1) can be written as follows:

$$\mathcal{L}(v) + \mathcal{N}(v) - f(t) = 0. \qquad \ldots\ldots\ldots\ (2)$$

Using homotopy technique, for a function

$\psi :\ \Omega \times [0,1] \to \mathbb{R}$, we define a homotopy

$\widetilde{H}(\psi, p) : \mathbb{R} \times [0,1] \to \mathbb{R}$ by

$$\widetilde{H}(\psi(t;p), p) = (1-p)\left[\mathcal{L}(\psi(t;p)) - \mathcal{L}(v_0(t))\right] + p[\mathcal{L}(\psi(t;p)) + \mathcal{N}(\psi(t;p)) - f(t)], \quad \ldots\ldots\ldots (3)$$

where $p \in [0,1]$ is a homotopy parameter, and ψ is a function of t and p.

In [33], Dr. Shijun Liao defined a new type of homotopy $H(\psi, p) : \mathbb{R} \times [0,1] \to \mathbb{R}$ by introducing a auxiliary parameter $\hbar$ and a auxiliary function $\mathcal{H}(t)$ so that $\hbar \neq 0$ and $\mathcal{H}(t) \neq 0$ as

$$H(\psi, p, \hbar, \mathcal{H}) = (1-p)\left[\mathcal{L}(\psi(t;p), p, \hbar, \mathcal{H}(t)) - \mathcal{L}(v_0(t))\right] - p\hbar\, \mathcal{H}(t)[\mathcal{L}(\psi(t;p), p, \hbar, \mathcal{H}(t)) + \mathcal{N}(\psi(t;p), p, \hbar, \mathcal{H}(t)) - f(t)] \quad \ldots\ \ldots\ \ldots (4)$$

Clearly (4) is more general than (3), since (3) is the special case of (4) for $\hbar = -1$ & $\mathcal{H}(t) = 1$. i.e., $\widetilde{H}(\psi(t,p), p) = H(\psi(t,p), p, -1, 1)$.

Then the deformation equation of order zero constructed by Liao [33] is given by

$$H[\psi(t;p); p, \hbar, \mathcal{H}] = (1-p)\mathcal{L}[\psi(t;p) - v_0(t)] - p\,\hbar\,\mathcal{H}(t)\mathcal{N}[\psi(t;p)], \qquad \ldots\ \ldots\ \ldots\ (5)$$

where p denotes the homotopy parameter with $p \in [0,1]$, $\mathcal{L}$ denotes the auxiliary linear operator with the property that

$$\mathcal{L}(0) = 0, \qquad \ldots\ \ldots\ \ldots\ (6)$$

$v_0(t)$ denotes the initial approximate solution, and $\mathcal{N}$ denotes the nonlinear operator which is defined as:

$$\mathcal{N}[\psi(t;p)] = \mathcal{L}[\psi(t;p)] + \mathcal{N}[\psi(t;p)] - f(t).$$

Substituting $H[\psi(t;p); p, \hbar, \mathcal{H}] = 0$ in the equation (5) the zeroth order deformation equation can be written as:

$$(1-p)\,\mathcal{L}[\psi(t;p) - v_0(t)] = p\,\hbar\,\mathcal{H}(t)\mathcal{N}[\psi(t;p)]. \quad \ldots\ \ldots\ (7)$$

If $p = 0$, then we have

$$H[\psi(t;p); p, \hbar, \mathcal{H}]|_{p=0} = \mathcal{L}[\psi(t;0) - v_0(t)] = 0$$

$$\text{i.e., } \mathcal{L}[\psi(t;0)] = v_0(t) \qquad \ldots\ \ldots\ \ldots\ (8)$$

and if $p = 1$, then we have

$$H[\psi(t;p); p, \hbar, \mathcal{H}]|_{p=1} = \hbar\,\mathcal{H}(t)\mathcal{N}[\psi(t;\ 1)] = 0$$

$$\text{i.e., } \hbar\,\mathcal{H}(t)\mathcal{N}[\psi(t;\ 1)] = 0 \qquad \ldots\ \ldots\ \ldots\ (9)$$

Now, it is clear from the equations (6), (8), and (9) that

$$\psi(t;0) = v_0(t) \text{ and } \psi(t;1) = v(t).$$

Therefore if the homotopy parameter p increases from 0 to 1, the solution $\psi(t;p)$ continuously changes from $v_0(t)$ to the solution $v(t)$ of the given equation (1). In topology, this type of continuous transformation is known as deformation.

Now, differentiating equation (7) m times w. r. to the homotopy parameter p, and putting $p = 0$, and finally multiplying them by $\frac{1}{m!}$, we get the deformation equation of order m as follows:

$$\mathcal{L}\,[v_m(t) - \mathcal{X}_m v_{m-1}(t)] = \hbar\,\mathcal{H}(t) D_{m-1}(\bar{v}_m), \quad \ldots\ \ldots\ \ldots\ (10)$$

where $\bar{v}_m = \{v_0(t), v_1(t), \ldots, v_m(t)\}$,

$$D_{m-1}(\bar{v}_m) = \frac{1}{(m-1)!}\ \frac{\partial^{m-1}\mathcal{N}[\psi(t;p)]}{\partial p^{m-1}} \quad \text{and}$$

$$\mathcal{X}_m = \begin{cases} 0, & \text{when } m \leq 1 \\ 1, & \text{when } m > 1 \end{cases}.$$

Since $\psi(t;p)$ depends on $p \in [0,1]$, by Taylor's theorem we have the series expansion of $\psi(t;p)$ w. r. to p as

$$\psi(t;p) = v_0(t) + \sum_{m=1}^{+\infty} v_m(t)\,p^m, \qquad \ldots\ldots\ldots (11)$$

where $v_m(t) = \frac{1}{m!}\frac{\partial^m \psi(t;p)}{\partial p^m}\Big|_{p=0}$.

Solving equation (10) we can find $v_m(t)$. If the initial approximation $v_0(t)$, the auxiliary linear operator $\mathcal{L}$, the auxiliary function $\mathcal{H}(t)$ and the auxiliary parameter ħ can be chosen properly, then the above series (11) must be convergent at $p = 1$.

Then at $p = 1$ the series (11) becomes

$$\psi(t;1) = v_0(t) + \sum_{\mathrm{m}=1}^{+\infty} v_m(t)$$

Therefore we have

$$v(t) = v_0(t) + \sum_{\mathrm{m}=1}^{+\infty} v_m(t). \qquad \dots\dots\dots (12)$$

In [33] Liao proved that the series (12) is one of the results of the given equation (1). It should be very significant to assure that at $p = 1$, the series (11) must be convergent, on the other hand there is no meaning of the series (12).

2.3 The Homotopy Perturbation Method (HPM):

To demonstrate the fundamental concepts of the HPM, we consider a differential equation, which is given by

$$\mathcal{A}(u) = f(r),\ r \in \Omega, \qquad \dots\dots\dots (13)$$

together with

$$\mathcal{B}\left(u, \frac{\partial u}{\partial m}\right) = 0,\ r \in \Gamma, \qquad \dots\dots\dots (14)$$

where $\mathcal{A}$ denotes the usual differential operator, $\mathcal{B}$ denotes the usual boundary operator, $f(r)$ denotes a known analytic function, the domain denoted by Ω, and Γ denotes the boundary of domain Ω.

Similar to HAM the usual differential operator $\mathcal{A}$ can be separated into two parts as $\mathcal{A}(u) = \mathcal{L}(u) + \mathcal{N}(u)$, where $\mathcal{L}$ stands for the linear operator, and $\mathcal{N}$ stands for non-linear operator in the given differential equation. Therefore equation (13) can be written as

$$\mathcal{L}(u) + \mathcal{N}(u) - f(r) = 0 \qquad \dots\dots\dots (15)$$

Using homotopy technique, we can define a homotopy as $w(r,p)\colon \Omega \times [0,1] \to \mathbb{R}$ and $H(w,p)\colon \mathbb{R} \times [0,1] \to \mathbb{R}$ satisfying the homotopy equation:

$$H(w,p) = (1-p)[\mathcal{L}(w) - \mathcal{L}(u_0)] + p[\mathcal{L}(w) + \mathcal{N}(w) - f(r)]$$
$$= 0, \quad p \in [0,1], r \in \Omega, \qquad \dots\dots\dots (16)$$

i.e.,

$$H(w,p) = \mathcal{L}(w) - \mathcal{L}(u_0) + p\,\mathcal{L}(u_0) + p[\mathcal{N}(w) - f(r)] = 0,$$
$$p \in [0,1], r \in \Omega, \qquad \dots\dots\dots (17)$$

where $p \in [0,1]$ denotes a homotopy parameter and u_0 denotes an initial approximate solution of the given differential equation (13) satisfying the given boundary conditions.

From equation (16) and (17), we have,

$$H(w,0) = \mathcal{L}(w) - \mathcal{L}(u_0) = 0;$$
$$H(w,1) = \mathcal{L}(w) + \mathcal{N}(w) - f(r) = 0.$$

The changing procedure of p from 0 (zero) to 1 (unity) is only that $w(r,p)$ shifting from $u_0(r)$ into $u(r)$, this is said to be homotopy, in topology. Therefore, we have,

$$\mathcal{L}(w) - \mathcal{L}(u_0) \cong \mathcal{L}(w) + \mathcal{N}(w) - f(r),\ r \in \Omega, \ \dots\dots\dots (18)$$

and $\quad w_0(r) \cong w(r), \quad r \in \Omega \qquad \dots\dots\dots (19)$

In topology, (19) is called deformation. Since $w(r,p)$ depends on homotopy parameter $p \in [0,1]$, by Taylor's theorem we have the series expansion of $w(r;\ p)$ w. r. to p as follows:

$$w = p^0 w_0 + p^1 w_1 + p^2 w_2 + p^3 w_3 + \cdots, \quad \dots\dots\dots (20)$$

i.e.,

$$w(r,p) = \sum_{\mathrm{i}=0}^{+\infty} p^i w_i(r).$$

Consider that this series expansion (20) gives the solutions of the equations (16) and (17).

Setting $p = 1$ in the equation (20), we get

$$w(r,1) = w_0 + w_1 + w_2 + w_3 + \cdots$$

Therefore

$$u(r) = \lim_{p\to1} w(r,p) = w_0 + w_1 + w_2 + w_3 + \cdots, \qquad \dots\dots\dots (21)$$

i.e.,

$$u(r) = \lim_{p\to1} w(r,p) = \sum_{\mathrm{i}=0}^{+\infty} w_i(r),$$

which is the solution of the given equation (13).

The perturbation method coupling with the homotopy technique is known as the homotopy perturbation method. This removes the constraint of the usual perturbation method. However, HPM has the full amenities of the usual perturbation method. For most cases, the series (21) is convergent.
However, to trace the rate of convergence on the non-linear operator, the following suggestions have been made by Dr. J-H. He [18]:

a) The $\frac{\partial^2 \mathcal{N}(w)}{\partial w^2}$ must be minimal so the parameter could be quite large, i.e., $p \to 1$.
b) The norm $\left\| \mathcal{L}^{-1} \frac{\partial \mathcal{N}}{\partial w} \right\| < 1$ so that the series converges.

3 NUMERICAL SCHEME OF THE K-dV EQUATION:

3.1 Numerical Scheme by the HAM: Consider the K-dV equation (a) with the conferred initial approximation

$$u(x,0) = 2sech^2(x).$$

Then the m$^{\text{th}}$ order deformation equation is

$$\mathcal{L}[u_m(x,t) - \mathcal{X}_m u_{m-1}(x,t)] = \hbar \mathcal{H}(x,t) D_{m-1}\big[\mathcal{N}\big(u(x,t)\big)\big], \qquad \dots\dots\dots (22)$$

where $\mathcal{L}(u) = \frac{\partial \mathrm{u}}{\partial \mathrm{t}}$, $\mathcal{N}(u) = \frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}$,

$$\mathcal{X}_m = \begin{cases} 0, & \text{when m} \le 1 \\ 1, & \text{when m} > 1 \end{cases} \quad \text{and} \quad u_0 = u(x,0) = 2sech^2(x).$$

Then we have,

$$(u_m)_t - \mathcal{X}_m (u_{m-1})_t = \hbar \mathcal{H} D_{m-1}[\mathcal{N}(u(x,t))].$$

Integrating both sides from 0 to 1 with respect to t, we get,

$$u_m - \mathcal{X}_m u_{m-1} = \int_0^t \hbar\mathcal{H} D_{m-1}[\mathcal{N}(u(x,t))]\, \mathrm{dt}$$

$$\Rightarrow u_m = \mathcal{X}_m u_{m-1} + \int_0^t \hbar\mathcal{H} D_{m-1}[\mathcal{N}(u(x,t))]\, \mathrm{dt} \quad \dots\dots\dots (23)$$

Now, $D_{m-1}[\mathcal{N}(u(x,t))] = D_{m-1}[u_t + 6uu_x + u_{xxx}]$

$= D_{m-1}[u_t] + D_{m-1}[6uu_x] + D_{m-1}[u_{xxx}]$

$=(u_{m-1})_t + 6\sum_{i=0}^{m-1} u_i\,.(u_{m-1-i})_x + (u_{m-1})_{xxx}$,

where $D_{m-1}[6uu_x] = 6\sum_{i=0}^{m-1} u_i\,.(u_{m-1-i})_x$.

Then the equation (23) can be written as

$$u_m = \mathcal{X}_m u_{m-1} + \hbar\mathcal{H}\int_0^t \Big[(u_{m-1})_t + 6\sum_{i=0}^{m-1} u_i\,.(u_{m-1-i})_x + (u_{m-1})_{xxx}\Big]\, dt \quad \dots\dots\dots (24)$$

Putting $m = 1, 2, 3, \cdots$, respectively in the above equation (24), we can find $u_1, u_2, u_3, \cdots$, as follows:

$$u_1 = -2^4.(\hbar.\mathcal{H}).t.sech^4(x)\,tanh(x) - 2^4.(\hbar.\mathcal{H}).t.sech^2(x).tanh^3(x).$$

$$u_2 = -2^4.\big((\hbar.\mathcal{H}) + (\hbar.\mathcal{H})^2\big).t.sech^4(x)\,tanh(x) - 2^4((\hbar.\mathcal{H}) + (\hbar.\mathcal{H})^2).t.sech^2(x)tanh^3(x) - 2^5.(\hbar.\mathcal{H})^2.t^2.sech^8(x) + 2^5.3(\hbar.\mathcal{H})^2.t^2.sech^4(x)\,tanh^4(x) + 2^6.(\hbar.\mathcal{H})^2.t^2.sech^2(x).tanh^6(x).$$

$$u_{3.} = (1 + (\hbar.\mathcal{H})).[-2^4.((\hbar.\mathcal{H}) + (\hbar.\mathcal{H})^2).t.sech^4(x)\,tanh(x) - 2^4.((\hbar.\mathcal{H}) + (\hbar.\mathcal{H})^2).t.\ sech^2(x).tanh^3(x) - 2^5(\hbar\mathcal{H})^2.t^2.sech^8(x) + 2^5.3.(\hbar.\mathcal{H})^2.t^2.sech^4(x)\,tanh^4(x) + 2^6.(\hbar.\mathcal{H})^2.t^2.sech^2(x).tanh^6(x)] + (\hbar.\mathcal{H}).[-2^5.((\hbar.\mathcal{H}) + (\hbar.\mathcal{H})^2).t^2.sech^8(x) + 2^5.3.\big((\hbar.\mathcal{H}) + (\hbar.\mathcal{H})^2\big).t^2.sech^4(x)\,tanh^4(x) + 2^6.\big((\hbar.\mathcal{H}) + (\hbar.\mathcal{H})^2\big).t^2.sech^2(x)\,tanh^6(x) + 2^{10}.(\hbar.\mathcal{H})^2.\frac{t^3}{3}.sech^{10}(x)tanh(x) + 2^9.5.(\hbar.\mathcal{H})^2.\frac{t^3}{3}.sech^8(x).tanh^3(x) - 2^9.(\hbar.\mathcal{H})^2.\frac{t^3}{3}.sech^4(x)\,tanh^7(x) + 2^9.3.(\hbar.\mathcal{H})^2.\frac{t^3}{3}.sech^6(x)\,tanh^5(x) - 2^9.(\hbar.\mathcal{H})^2.\frac{t^3}{3}.sech^2(x)\,tanh^9(x)].$$

$$u_4 = (1 + \hbar.\mathcal{H}).u_3 + (\hbar.\mathcal{H} + (\hbar.\mathcal{H})^2).[2^6.(\hbar.\mathcal{H}^2 + (\hbar.\mathcal{H})).t^2.\ sech^2(x)\,tanh^6(x) - 2^5.(\,\hbar.\mathcal{H} + (\hbar.\mathcal{H})^2).t^2.\ sech^8(x). + 2^5.3.(\hbar.\mathcal{H} + (\hbar.\mathcal{H})^2).t^2.sech^4(x)\,tanh^4(x) - 2^9.(\hbar.\mathcal{H})^2.\frac{t^3}{3}.sech^{10}(x).tanh(x) + 2^9.5.(\hbar.\mathcal{H})^2.\frac{t^3}{3}.sech^4(x)\,tanh^7(x) + 2^{11}.3.(\hbar.\mathcal{H})^2.\frac{t^3}{3}sech^6(x)\,tanh^5(x) + 2^9.5.(\hbar.\mathcal{H})^2.\frac{t^3}{3}.sech^8(x)\,tanh^3(x) - 2^9\,(\hbar.\mathcal{H})^2.\frac{t^3}{3}sech^2(x)\,tanh^9(x)] - 2^9.(\,(\hbar.\mathcal{H})^3 + (\hbar.\mathcal{H})^4)\frac{t^3}{3}.sech^2(x)\,.tanh^9(x) + 2^7.37.((\hbar.\mathcal{H})^3 + (\hbar.\mathcal{H})^4).\frac{t^3}{3}.sech^{10}(x)\,tanh(x) + 2^9\,.5.((\hbar.\mathcal{H})^3 + (\hbar.\mathcal{H})^4).\frac{t^3}{3}\,.sech^8(x).tanh^3(x) - 2^9.7.((\hbar.\mathcal{H})^3 + (\hbar.\mathcal{H})^4).\frac{t^3}{3}.sech^4(x)tanh^7(x) - 2^{10}\,3.((\hbar.\mathcal{H})^3 + (\hbar.\mathcal{H})^4).\frac{t^3}{3}.sech^6(x)\,tanh^5(x) - 2^{12}.3.5.(\hbar.\mathcal{H})^4.\frac{t^4}{12}.sech^{10}(x)\,tanh^4(x) - 2^{11}.3.(\hbar.\mathcal{H})^4.\frac{t^4}{12}.sech^{12}(x).tanh^2(x) + 2^{12}.(\hbar.\mathcal{H})^4.\frac{t^4}{12}.sech^{14}(x) - 2^{12}.3.5.(\hbar.\mathcal{H})^4.\frac{t^4}{12}sech^6(x)\,.tanh^8(x) - 2^{12}.5^2.(\hbar.\mathcal{H})^4.\frac{t^4}{12}.sech^8(x)\,tanh^6(x) - 2^{11}.3.(\hbar.\mathcal{H})^4.\frac{t^4}{12}.sech^4(x)\,tanh^{10}(x) + 2^{12}.(\hbar.\mathcal{H})^4.\frac{t^4}{12}.sech^2(x).tanh^{12}(x).$$

… … …

Therefore the solution series is

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots\cdots\cdots$$

$$= 2sech^2(x) - 2^4.(\hbar.\mathcal{H}).t.sech^4(x)\,tanh(x) - 2^4.(\hbar.\mathcal{H}).t.sech^2(x).tanh^3(x) - 2^4.\big((\hbar.\mathcal{H}) + (\hbar.\mathcal{H})^2\big).t.sech^4(x)\,tanh(x) - 2^4((\hbar.\mathcal{H}) + (\hbar.\mathcal{H})^2).t.sech^2(x)tanh^3(x) - 2^5.(\hbar.\mathcal{H})^2.t^2.sech^8(x) + 2^5.3(\hbar.\mathcal{H})^2.t^2.sech^4(x)\,tanh^4(x) + 2^6\,(\hbar.\mathcal{H})^2.t^2.sech^2(x).tanh^6(x) + (1 + (\hbar.\mathcal{H})).[-2^4.\big((\hbar.\mathcal{H}) + (\hbar.\mathcal{H})^2\big).t.sech^4(x)\,tanh(x) - 2^4.\big((\hbar.\mathcal{H}) + (\hbar.\mathcal{H})^2\big).t.sech^2(x).tanh^3(x) - 2^5.(\hbar.\mathcal{H})^2.t^2..sech^8(x) + 2^5.3.(\hbar.\mathcal{H})^2.t^2.sech^4(x)\,tanh^4(x) + 2^6.(\hbar.\mathcal{H})^2.t^2.sech^2(x).tanh^6(x)] + (\hbar.\mathcal{H}).[-2^5.((\hbar.\mathcal{H}) + (\hbar.\mathcal{H})^2).t^2.sech^8(x) + 2^5.3.\big((\hbar.\mathcal{H}) + (\hbar.\mathcal{H})^2\big).t^2.sech^4(x).tanh^4(x) + 2^6.\big((\hbar.\mathcal{H}) + (\hbar.\mathcal{H})^2\big).t^2.sech^2(x).tanh^6(x) + 2^{10}.(\hbar.\mathcal{H})^2\,.\frac{t^3}{3}\,.sech^{10}(x)\ tanh(x) + 2^9.5.(\hbar.\mathcal{H})^2\,.\frac{t^3}{3}\,.sech^8(x).tanh^3(x) - 2^9.(\hbar.\mathcal{H})^2.\frac{t^3}{3}.sech^4(x)\,tanh^7(x) + 2^9.3.(\hbar.\mathcal{H})^2.\frac{t^3}{3}.sech^6(x)\,tanh^5(x) - 2^9.(\hbar.\mathcal{H})^2.\frac{t^3}{3}.sech^2(x)\,tanh^9(x)] + (1 + (\hbar.\mathcal{H})).\ u_3 + (\hbar.\mathcal{H} + (\hbar.\mathcal{H})^2).[2^6.(\hbar.\mathcal{H} + (\hbar.\mathcal{H})^2).t^2.sech^2(x)\ tanh^6(x) - 2^5.(\hbar.\mathcal{H} + (\hbar.\mathcal{H})^2).t^2.sech^8(x) + 2^5.3$$

$$(\hbar.\mathcal{H}+(\hbar.\mathcal{H})^2).t^2.sech^4(x)\ tanh^4(x)-2^9.(\hbar.\mathcal{H})^2.\frac{t^3}{3}.$$
$$sech^{10}(x).tanh(x)+2^9.5.(\hbar.\mathcal{H})^2.\frac{t^3}{3}.\text{sech}^4(\text{x})\ \text{tanh}^7(\text{x})+2^{11}.3.$$
$$(\hbar.\mathcal{H})^2.\frac{t^3}{3}.sech^6(x)\ tanh^5(x)+2^9.5.(\hbar.\mathcal{H})^2.\frac{t^3}{3}.sech^8(x)$$
$$tanh^3(x)-2^9.(\hbar.\mathcal{H})^2.\frac{t^3}{3}.sech^2(x)\ tanh^9(x)]-2^9.((\hbar.\mathcal{H})^3+$$
$$(\hbar.\mathcal{H})^4).\frac{t^3}{3}.sech^2(x).tanh^9(x)+2^7.\ 37.((\hbar.\mathcal{H})^3+(\hbar.\mathcal{H})^4).$$
$$\frac{t^3}{3}.sech^{10}(x)\ tanh(x)+2^9.5.((\hbar.\mathcal{H})^3+(\hbar.\mathcal{H})^4).\frac{t^3}{3}.sech^8(x).$$
$$tanh^3(x)-2^9.7((\hbar.\mathcal{H})^3+(\hbar.\mathcal{H})^4).\frac{t^3}{3}.sech^4(x)\ tanh^7(x)-2^{10}.$$
$$3.((\hbar.\mathcal{H})^3+(\hbar.\mathcal{H})^4).\frac{t^3}{3}.sech^6(x)\ tanh^5(x)-2^{12}.3.5.(\hbar.\mathcal{H})^4.$$
$$\frac{t^4}{12}.sech^{10}(x)\ tanh^4(x)-2^{11}.3.(\hbar.\mathcal{H})^4.\frac{t^4}{12}.sech^{12}(x).tanh^2(x)$$
$$+\ 2^{12}.(\hbar.\mathcal{H})^4.\frac{t^4}{12}.\ sech^{14}(x)\ -2^{12}.3.5.(\hbar.\mathcal{H})^4.\frac{t^4}{12}sech^6(x).$$
$$tanh^8(x)-2^{12}.5^2.(\hbar.\mathcal{H})^4.\frac{t^4}{12}.\ sech^8(x)\ tanh^6(x)\ -2^{11}.3.$$
$$(\hbar.\mathcal{H})^4.\frac{t^4}{12}.sech^4(x)\ tanh^{10}(x)+2^{12}.(\hbar.\mathcal{H})^4.\frac{t^4}{12}.sech^2(x)$$
$$tanh^{12}(x)+\cdots\cdots\cdots, \qquad \ldots\ \ldots\ (25)$$

3.2 Numerical Scheme by the HPM:

For solving equation (a) by the HPM, we start by making a homotopy

$w:\Omega\times[0,1]\longrightarrow\mathbb{R}^2$, which satisfies the homotopy equation

$$H(w,p)=\mathcal{L}(w)-\mathcal{L}(u_0)+p\mathcal{L}(u_0)+p[\mathcal{N}(w)-f(x,t)]=0,$$

where $\mathcal{L}=\frac{\partial}{\partial t}$, $\mathcal{N}(w)=6ww_x+w_{xxx}$, $f(x,t)=0$ & $p\in[0,1]$.

Then we have, $w_t-(u_0)_t+p(u_0)_t+p[6ww_x+w_{xxx}]=0.$

$$\ldots\ \ldots\ \ldots\quad(26)$$

Substituting the initial condition in equation (26), we have,

$$w_t-(2sech^2(x))_t+p(2\,sech^2(x))_t+p[6ww_x+w_{xxx}]=0$$

i.e., $w_t-0+p.0+p[6ww_x+w_{xxx}]=0$

i.e., $w_t+p[6ww_x+w_{xxx}]=0.$ …… …. (27)

In equation (27), substituting $w=w_0+pw_1+p^2w_2+p^3w_3+\cdots$, we have

$$(w_0+pw_1+p^2w_2+p^3w_3+\cdots)+p[6.(w_0+pw_1+p^2w_2+p^3w_3+\cdots).(w_0+pw_1+p^2w_2+p^3w_3+\cdots)_x+(w_0+pw_1+p^2w_2+p^3w_3+\cdots)_{xxx} \qquad .\ ..\ \ldots\ldots\ (28)$$

For simplification we consider $u(x,0)=w(x,0)=2sech^2(x)$

i.e., $(w_0+pw_1+p^2w_2+p^3w_3+\cdots\cdots\cdots)(x,0)=2sech^2(x)$.

Which implies that

$w_0(x,0)=2sech^2(x)$; $w_1(x,0)=w_2(x,0)=w_3(x,0)=\cdots=0.$

Now, equation (28) can be written as

$$p^0(w_0)_t+p^1[(w_1)_t+6w_0(w_0)_x+(w_0)_{xxx}]+p^2[(w_2)_t+6w_0.(w_1)_x+6\,w_1(w_0)_x+(w_1)_{xxx}]+p^3[(w_3)_t+6\,w_2(w_0)_x+6\,w_1.(w_1)_x+6\,w_0\,(w_2)_x+(w_2)_{xxx}]+\cdots\cdots\cdots+p^n[(w_n)_t+6.\sum_{i=0}^{n-1}\text{w}_\text{i}\,(\text{w}_{\text{n}-1-\text{i}})_\text{x}+(w_{n-1})_{xxx}+\cdots\cdots\cdots=0.$$

This equation can be represented as:

p^0: $(w_0)_t=0$; $w_0(x,0)=2sech^2(x)$,

p^1: $(w_1)_t+6w_0(w_0)_x+(w_0)_{xxx}=0$; $w_1(x,0)=0$,

p^2: $(w_2)_t+6w_0(w_1)_x+6w_1(w_0)_x+(w_1)_{xxx}=0$; $w_2(x,0)=0$,

p^3: $(w_3)_t+6w_0\,(w_2)_x+6w_1\,(w_1)_x+6w_2\,(w_0)_x+(w_2)_{xxx}=0$; $w_3(x,0)=0$,

⋮

p^n: $(w_n)_t+6.\sum_{i=0}^{n-1}\text{w}_\text{i}\,(\text{w}_{\text{n}-1-\text{i}})_\text{x}+(w_{n-1})_{xxx}=0$; $w_n(x,0)=0$,

⋮

Solving the above equation we can find $w_0,w_1,w_2,w_3,\cdots$, as follows:

$$w_0=2sech^2(x);$$
$$w_1=2^4.t.sech^4(x).tanh(x)+2^4.t.sech^2(x).tanh^3(x);$$
$$w_2=2^5.3.t^2.sech^4(x).tanh^4(x)-2^5.t^2.sech^8(x)+2^6.t^2.sech^2(x).tanh^6(x);$$
$$w_3=-2^9.5.\frac{t^3}{3}.sech^8(x).tanh^3(x)-2^{10}.\frac{t^3}{3}.sech^{10}(x).tanh(x)+2^9.\frac{t^3}{3}.sech^4(x).tanh^7(x)-2^9.3.\frac{t^3}{3}.sech^6(x).tanh^5(x)+2^9.\frac{t^3}{3}.sech^2(x).tanh^9(x);$$
$$w_4=-2^{12}.3.5.\frac{t^4}{12}.\text{sech}^{10}(\text{x})\ \text{tanh}^4(\text{x})-2^{11}.3.\frac{t^4}{12}.\text{sech}^{12}(\text{x}).tanh^2(x)+2^{12}.\frac{t^4}{12}.sech^{14}(x)-2^{12}.3.5.\frac{t^4}{12}.sech^6(x).tanh^8(x)-2^{12}.5^2.\frac{t^4}{12}.sech^8(x)\ tanh^6(x)-2^{11}.3.\frac{t^4}{12}.sech^4(x)\ tanh^{10}(x)+2^{12}.\frac{t^4}{12}.sech^2(x)\ tanh^{12}(x);$$

… … …

As a result, the solution series becomes

$$u(x,t)=\lim_{\text{p}\to1}w(x,t,p)=w_0+w_1+w_2+w_3+\cdots\cdots\cdots$$
$$=2sech^2(x)+2^4\,t\,sech^4(x)\,tanh(x)+2^4.t\,sech^2(x)\ tanh^3(x)$$
$$+2^5.3.t^2.sech^4(x).\ tanh^4(x)-2^5.t^2.\ sech^8(x)\ +2^6.t^2.$$

$$sech^2(x).\ tanh^6(x)\ - 2^9.5.\ \frac{t^3}{3}.sech^8(x).\ tanh^3(x) - 2^{10}.\frac{t^3}{3}.$$
$$sech^{10}(\mathrm{x}).\tanh(\mathrm{x}) + 2^9.\frac{t^3}{3}.sech^4(x).tanh^7(x) - 2^9.3.\frac{t^3}{3}.$$
$$sech^6(x)\,.tanh^5(x)\ + 2^9.\frac{t^3}{3}.\ sech^2(x).\ tanh^9(x)\ - 2^{12}.3.5.$$
$$\frac{t^4}{12}.sech^{10}(x)\ tanh^4(x) - 2^{11}.3\frac{t^4}{12}sech^{12}(x).tanh^2(x) + 2^{12}.\frac{t^4}{12}$$
$$sech^{14}(x) - 2^{12}.3.5\frac{\mathrm{t}^4}{12}sech^6(x)\ tanh^8(x) - 2^{12}.5^2.\frac{t^4}{12}.sech^8(x)$$
$$.tanh^6(x) - 2^{11}.3.\ \frac{\mathrm{t}^4}{12}.\ sech^4(x)\ tanh^{10}(x) + 2^{12}.\ \frac{t^4}{12}\ sech^2(x).$$
$$tanh^{12}(x)\ + \cdots\cdots\cdots \qquad \ldots\ldots\ldots (29)$$

4 Results and Discussion:

We have computed the results of the K-dV equation by both HAM and HPM using our MATLAB routine and have compared the results with the exact solution in the interval $[-5,5]$. In this case we have considered $c = 4$ in the exact solution

$$u(x,t) = \frac{c}{2}\ sech^2(\frac{\sqrt{c}}{2}\ (x - ct)) \qquad \ldots\ldots\ldots (b).$$

It is to be noted here that we have considered three cases for both HAM and HPM. Case-I is with three terms $u_0,\ u_1,\ u_2$ for HAM, and $w_0,\ w_1,\ w_2$ for HPM. Case-II is with four terms $u_0,\ u_1,\ u_2,\ u_3$ for HAM, and $w_0,\ w_1,\ w_2,\ \ w_3$ for HPM. And Case-III is with five terms $u_0,\ u_1,\ u_2,\ u_3,\ u_4$ for HAM, and $w_0,\ w_1,\ w_2,\ w_3,\ w_4$ for HPM.

4.1 Discussion of the Numerical Results:

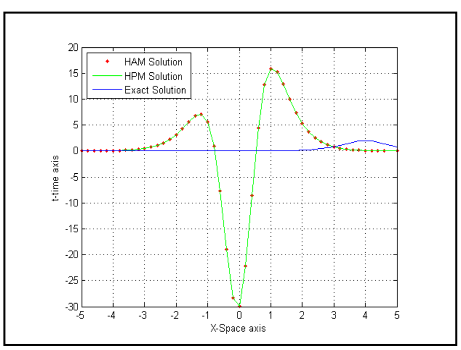
Fig. 1 shows that for the product of auxiliary parameter($\hbar$) and auxiliary function($\mathcal{H}$(t)) equal to-1, i.e., $\hbar\mathcal{H} = -1$, the solution of the K-dV equation (a) obtained by HAM coincides with the solution obtained by HPM and both solutions agree with the exact solution (b) for all the three cases.

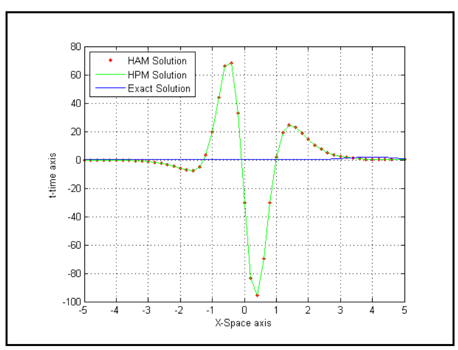
Figs. 2, 3, 5, and 6 show that for $(\mathcal{H},\hbar) = (-1,0.5),\ (-1,-0.5),$ $(-1,0.25), (-0.5,-1), (-0.5,-0.5), (-0.5\ ,0.5), (-0.5,1),$ $(-0.25,-1), (-0.25,1), (0.5,-1), (0.5,-0.5)(0.5,0.5), (0.5,1),$ $(1,-0.5),\ (1,0.25)$ and $(1,0.5)$, HAM gives better solution for large values of time t where as HPM gives more accurate results for small values of t and results from both HAM and HPM agree well with the exact solution (b) for all the three cases.

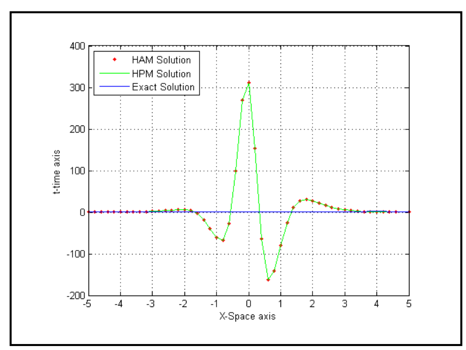
Fig. 4 shows that for $(\mathcal{H},\hbar) = (-1,-1), (1,1), \left(-2,-\frac{1}{2}\right),$ $\left(\frac{1}{2},2\right), \left(3,\frac{1}{3}\right),$ and $\left(-\frac{1}{3},-3\right)$ HPM solution is more accurate than HAM for all values of t for all the three cases. It is to be noted here that for small values of t results from both HAM and HPM agree well with the exact solution (b) for all the three cases.

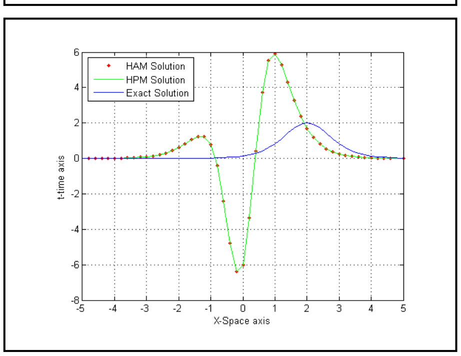
4.2 Error Analysis:

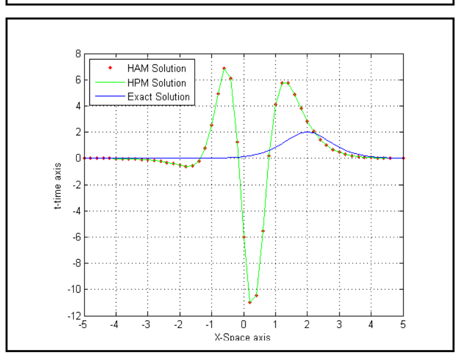
Using our MATLAB routine, we have been computed l_2-errors of the solution of K-dV equation (a) obtained by both HAM and HPM for all the three cases in different values of time t using the exact solution (b) and the error formula, error $= \sqrt{\sum_{i=1}^{n}(f_e(x_i) - f_c(x_i))^2}$, where f_{e}, f_{c} are exact and computed solutions, respectively. The results represented in Table $(1-6)$. The results show that for smaller values of time t the error decreases. For all the cases HPM has minimum error for very small values of time t.

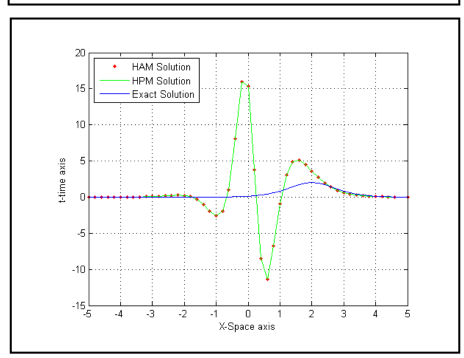












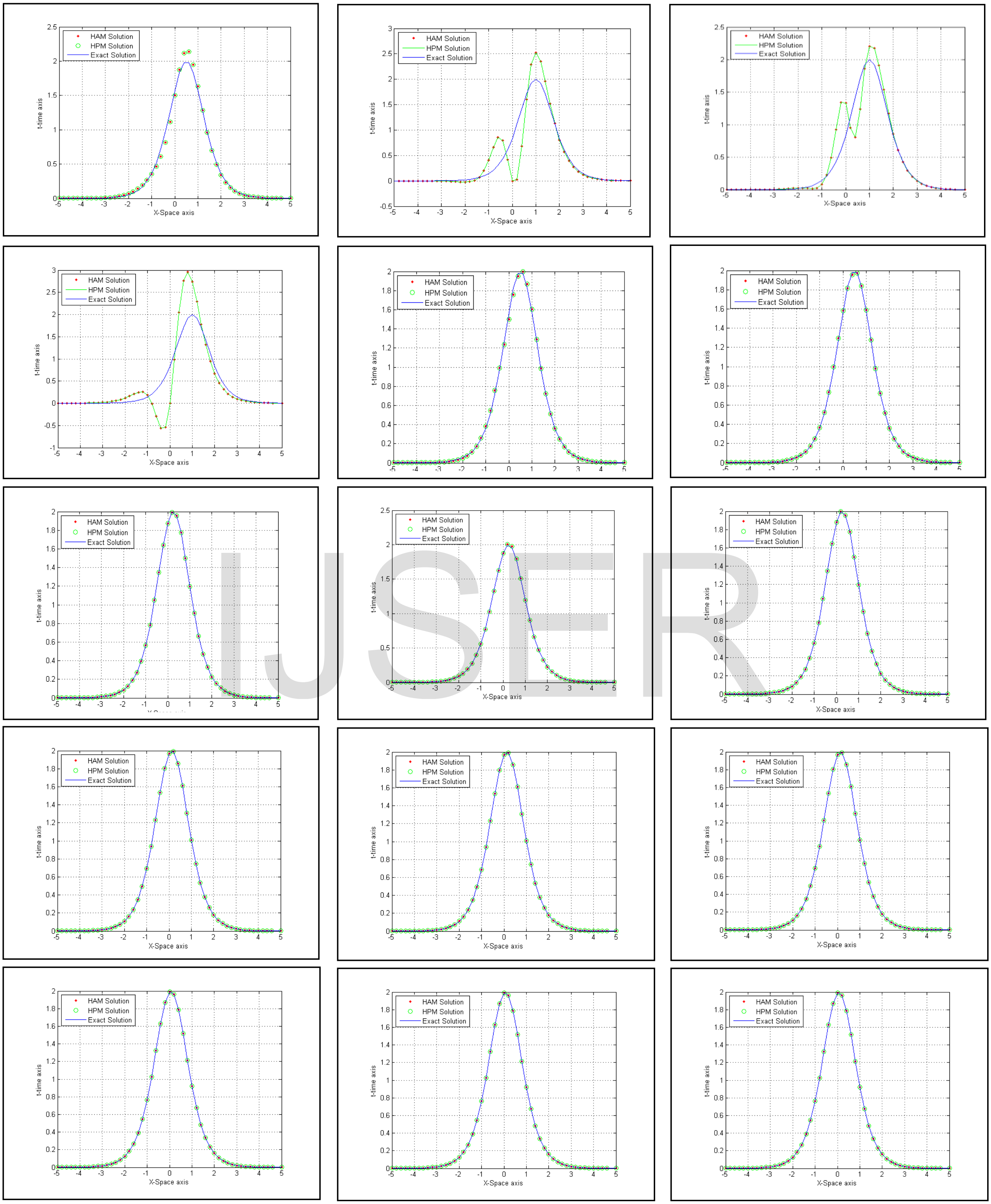
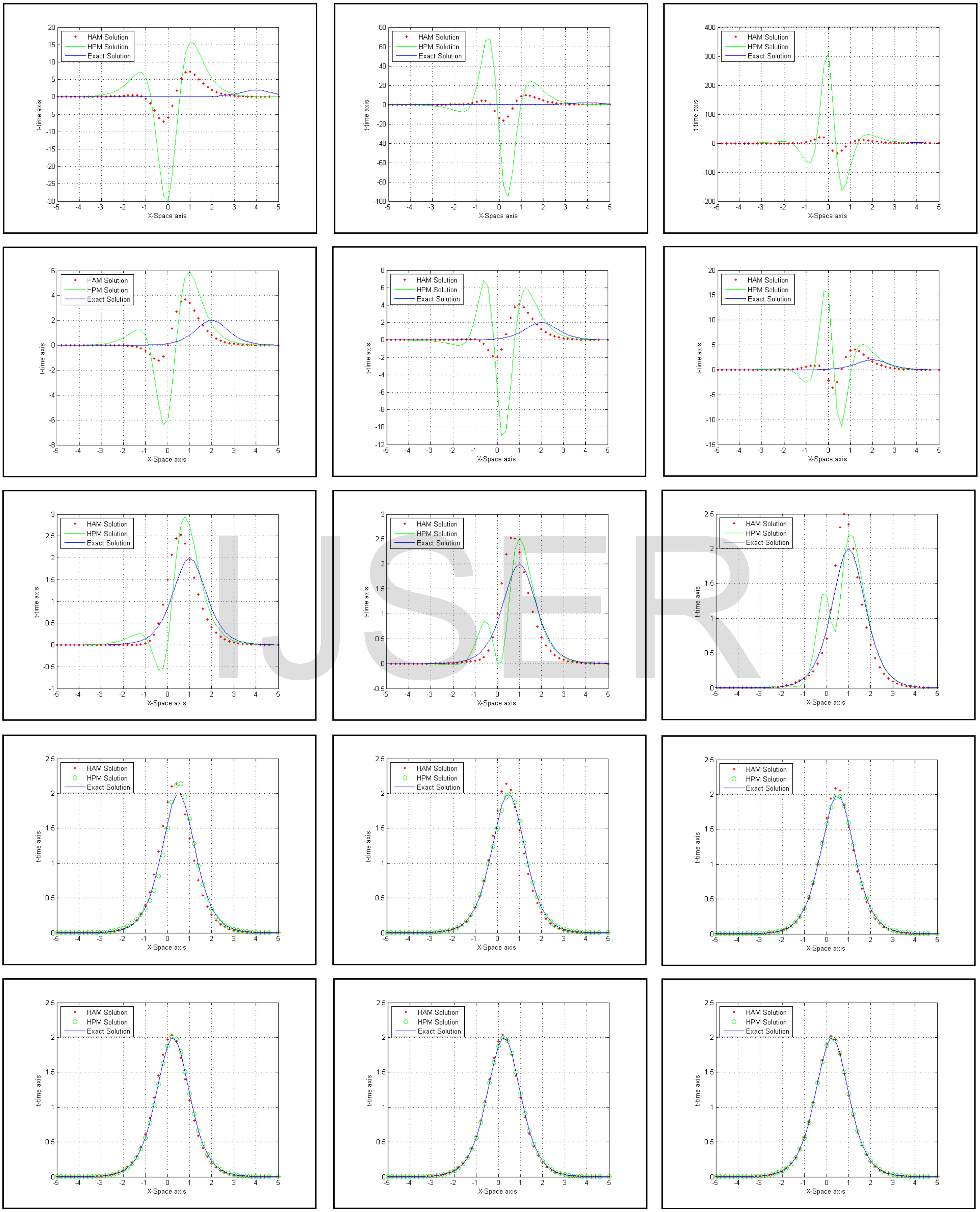


Fig. 1: 1st, 2nd and 3rd columns represent the exact solution, HAM solution and HPM solution of the K-dV equation for Case-I, Case-II and Case-III, respectively with regard to the values $(\mathcal{H},\hbar) = (-1\,,1), (1,-1), \left(-2,\frac{1}{2}\right), \left(-\frac{1}{2},2\right), \left(3,-\frac{1}{3}\right)$, and $\left(\frac{1}{3},-3\right)$ in HAM.


HAM Solution
HPM Solution
Exact Solution
t-time axis
X-Space axis

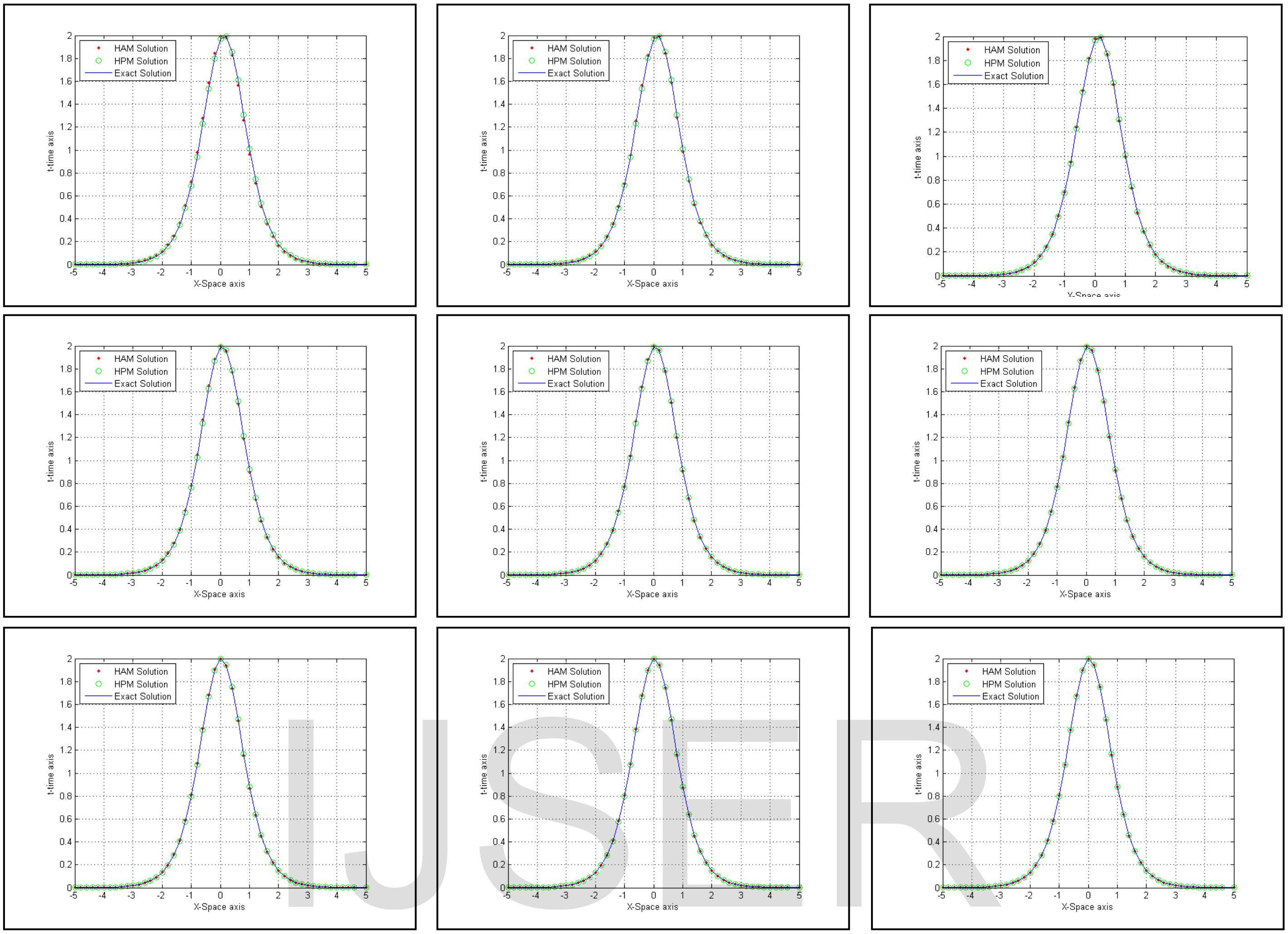
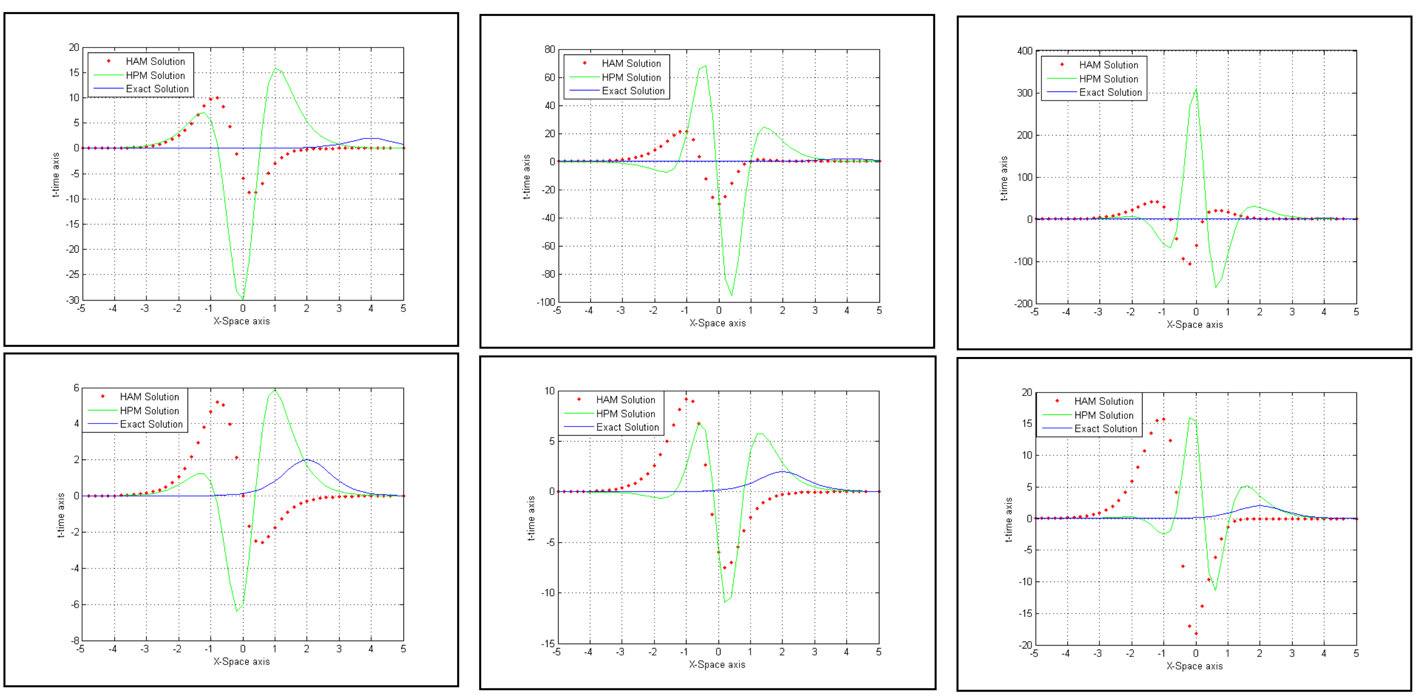
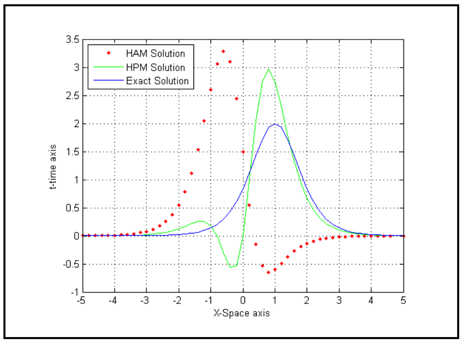


Fig. 2: 1st, 2nd and 3rd columns represent the exact solution, HAM solution and HPM solution of the K-dV equation for Case-I, Case-II and Case-III, respectively with regard to the values $(\mathcal{H}, \hbar) = (-1\,, 0.5), (1, -0.5), (0.5, -1)$, and $(-0.5, 1)$ in HAM.

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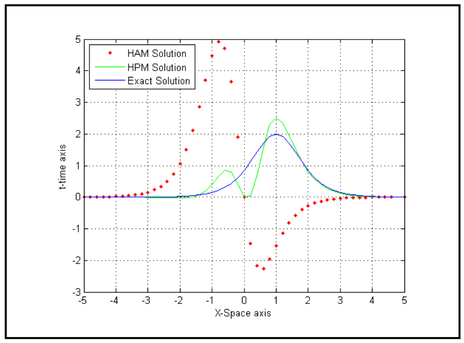
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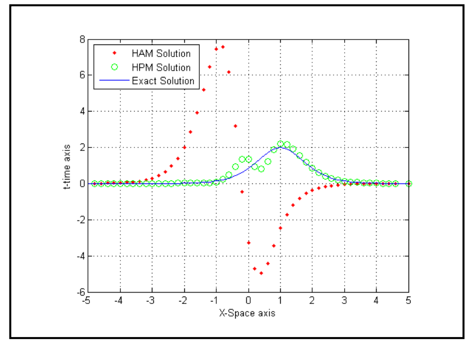
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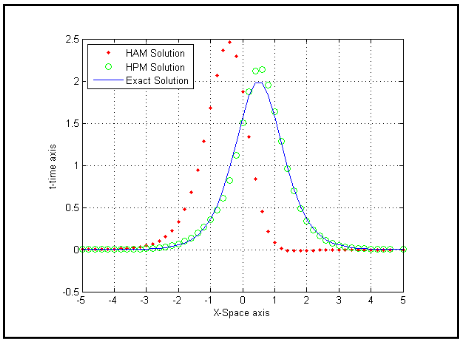
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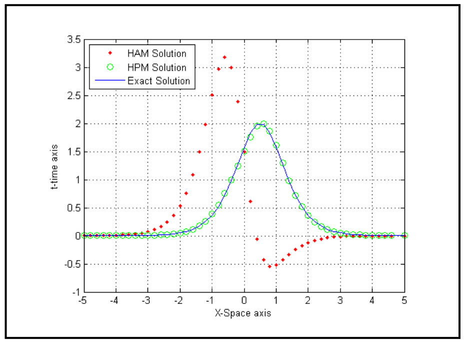
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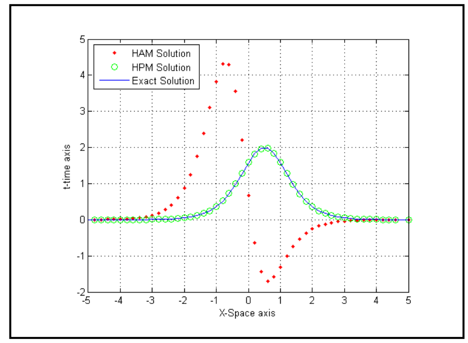
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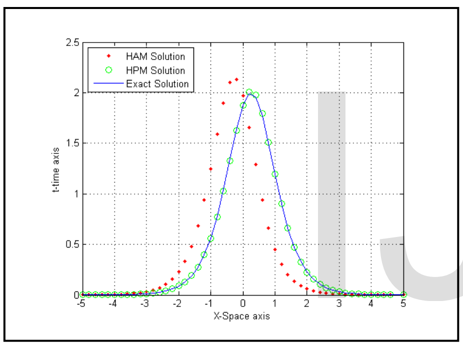
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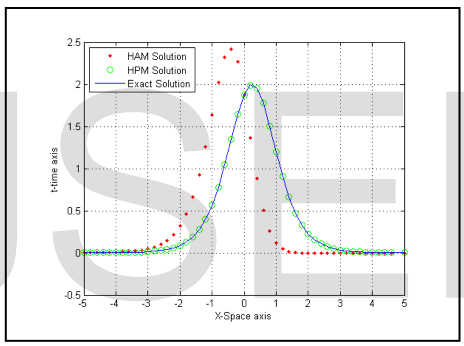
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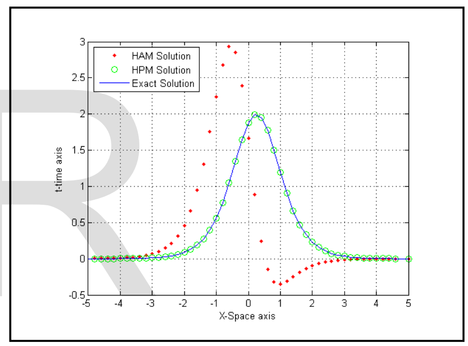
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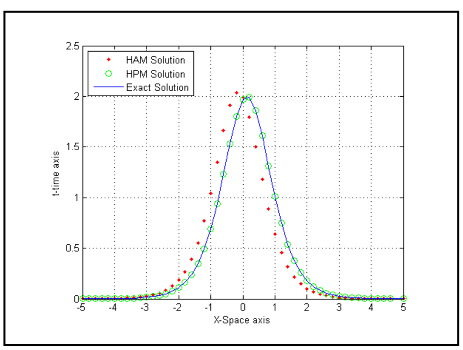
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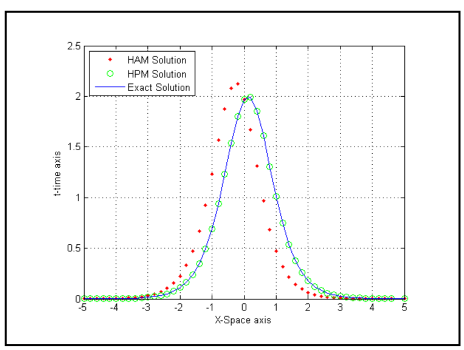
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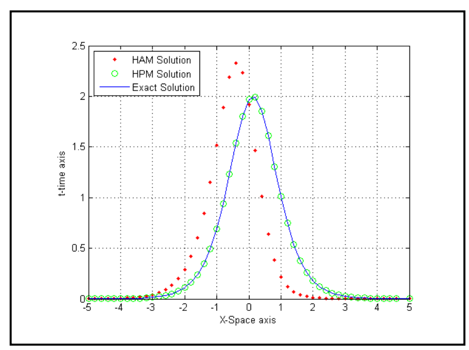
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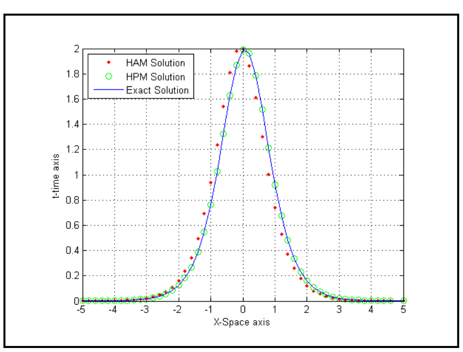
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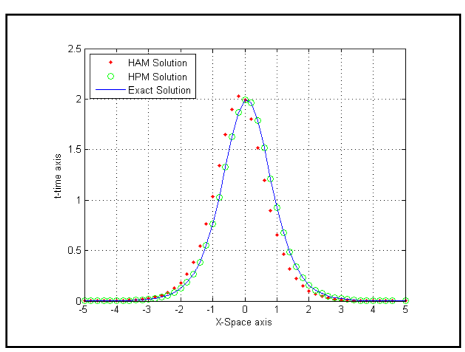
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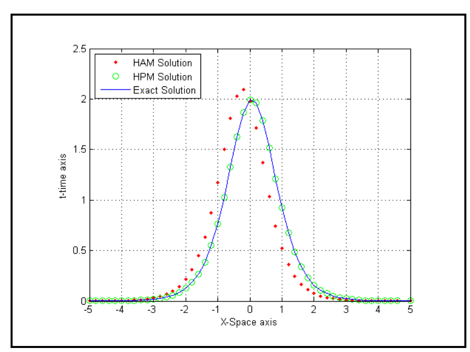
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X-Space axis

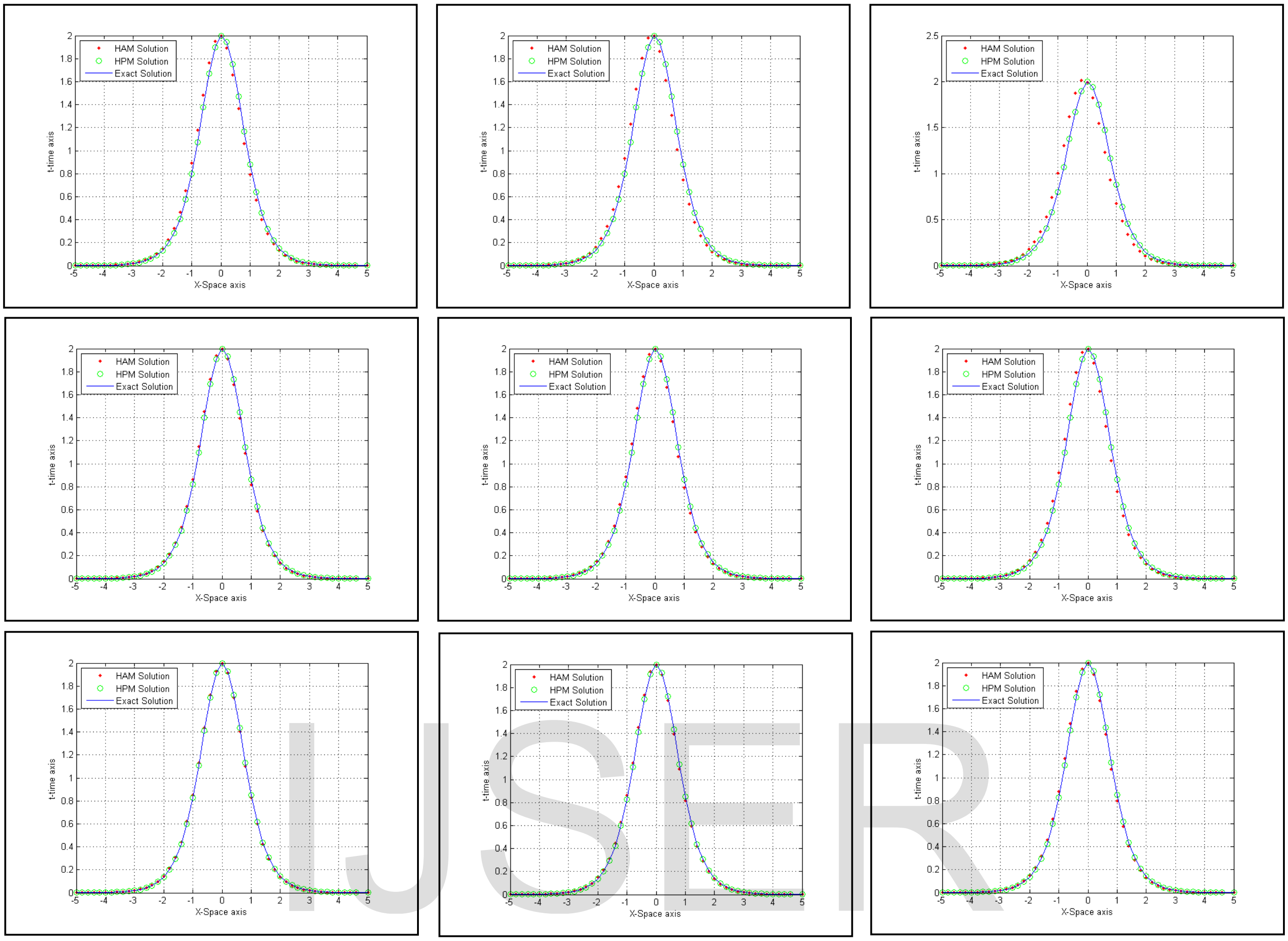
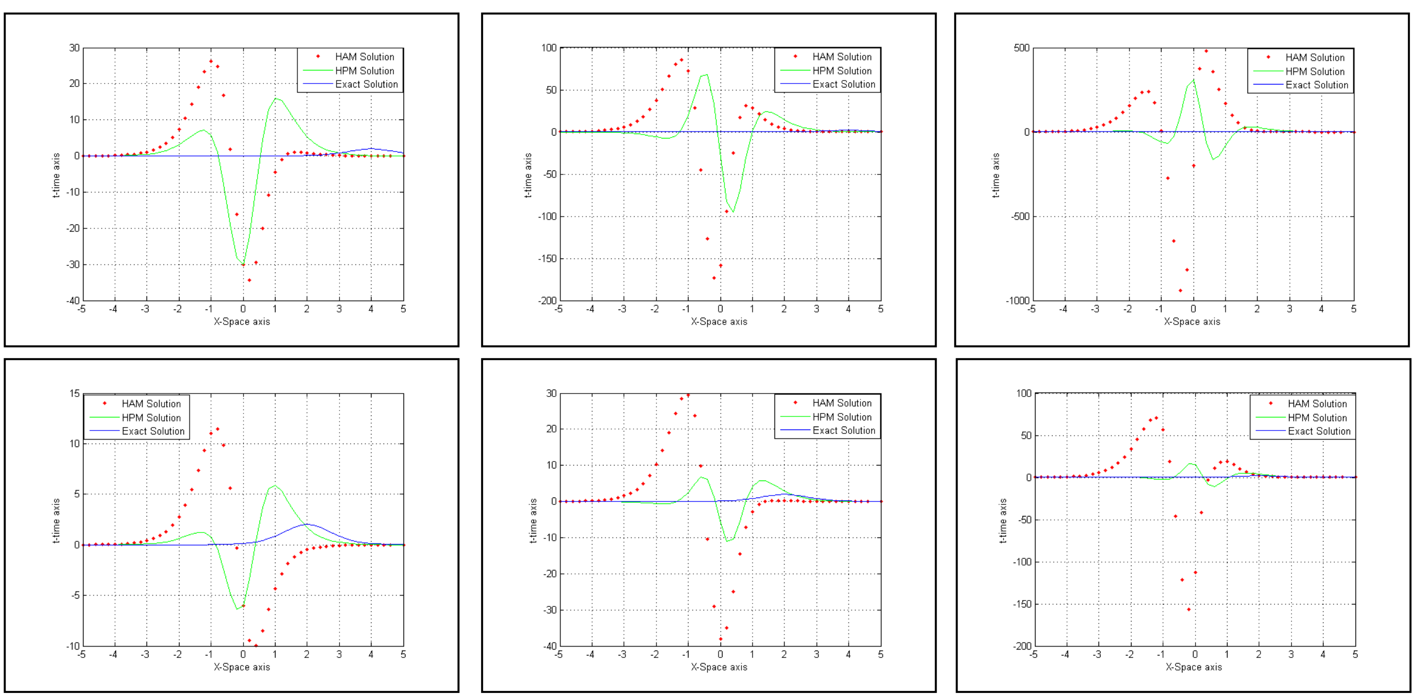
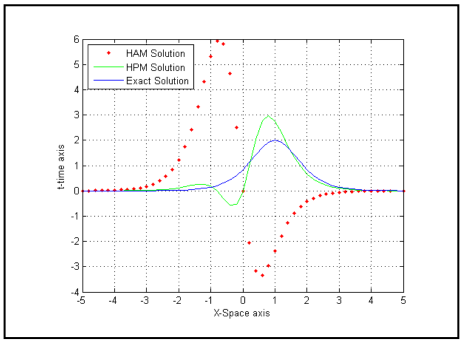
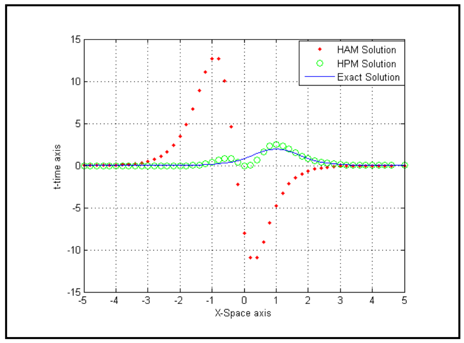
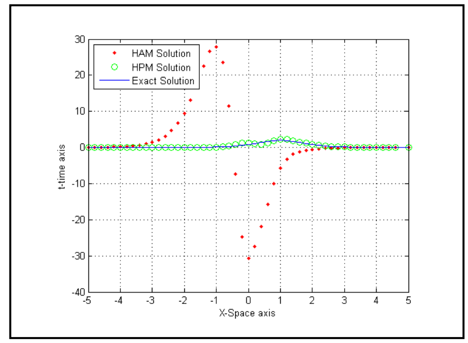


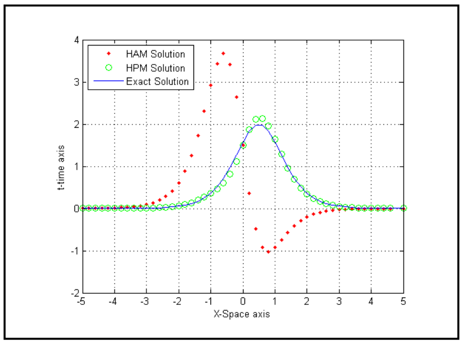
Fig. 3: 1st, 2nd and 3rd columns represent the exact solution, HAM solution and HPM solution of the K-dV equation for Case-I, Case-II and Case-III, respectively with regard to the values $(\mathcal{H}, \hbar) = (-1\,, -0.5), (-0.5, -1), (1, 0.5)$, and $(0.5, 1)$ in HAM.




HAM Solution
HPM Solution
Exact Solution
t-time axis
X-Space axis


HAM Solution
HPM Solution
Exact Solution
t-time axis
X-Space axis


HAM Solution
HPM Solution
Exact Solution
t-time axis
X-Space axis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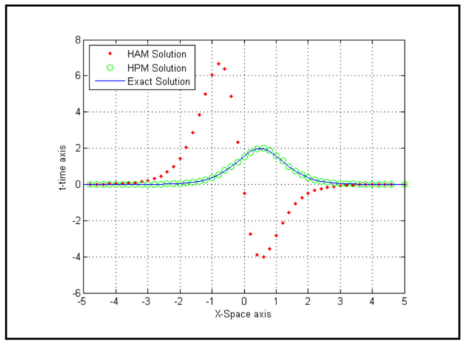
HPM Solution
Exact Solution
t-time axis
X-Space axis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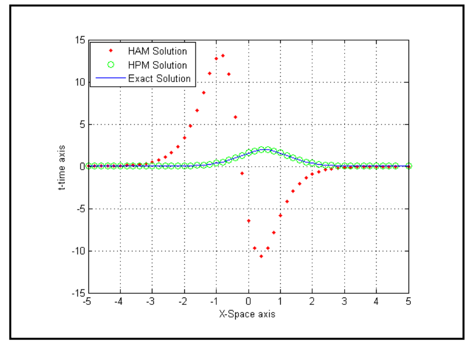
HPM Solution
Exact Solution
t-time axis
X-Space axis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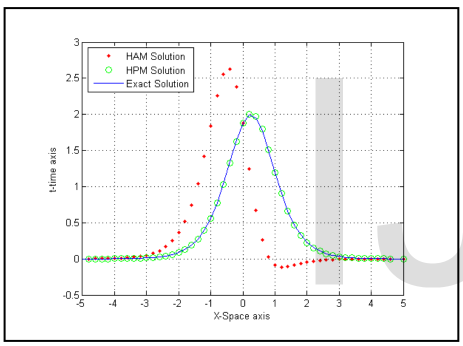
HPM Solution
Exact Solution
t-time axis
X-Space axis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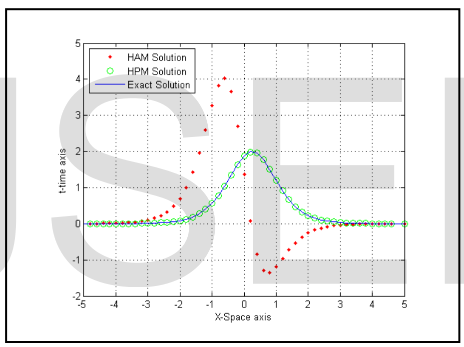
HPM Solution
Exact Solution
t-time axis
X-Space axis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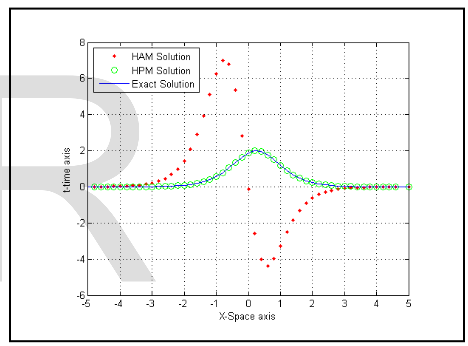
HPM Solution
Exact Solution
t-time axis
X-Space axis


HAM Solution
HPM Solution
Exact Solution
t-time axis
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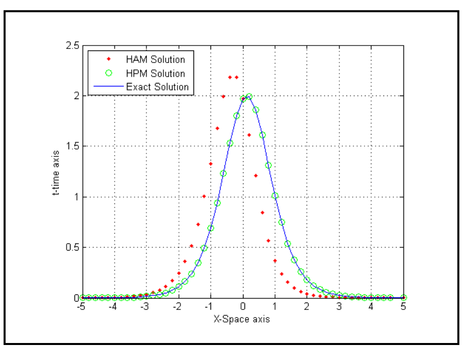
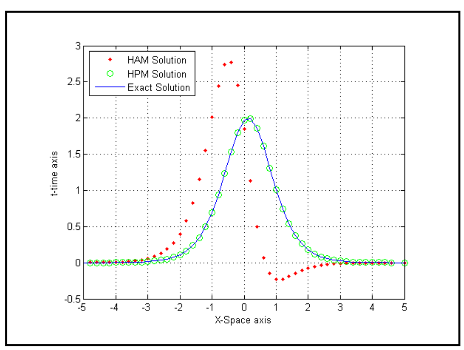

HAM Solution
HPM Solution
Exact Solution
t-time axis
X-Space axis


HAM Solution
HPM Solution
Exact Solution
t-time axis
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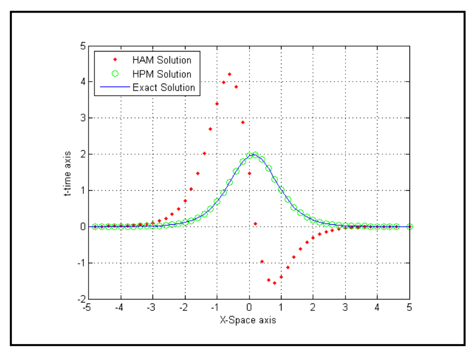
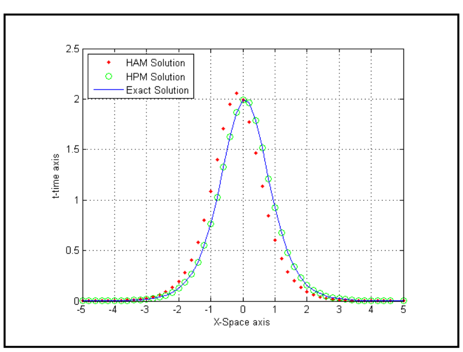

HAM Solution
HPM Solution
Exact Solution
t-time axis
X-Space axis


HAM Solution
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Exact Solution
t-time axis
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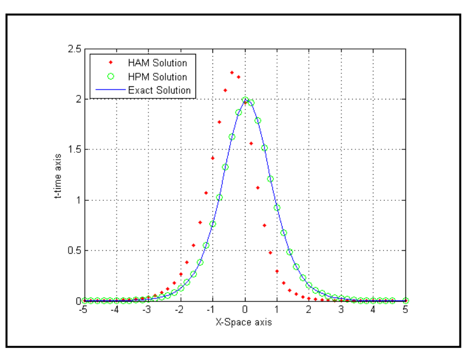
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t-time axis
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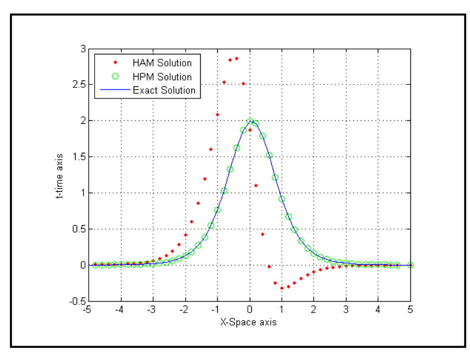

HAM Solution
HPM Solution
Exact Solution
t-time axis
X-Space axis

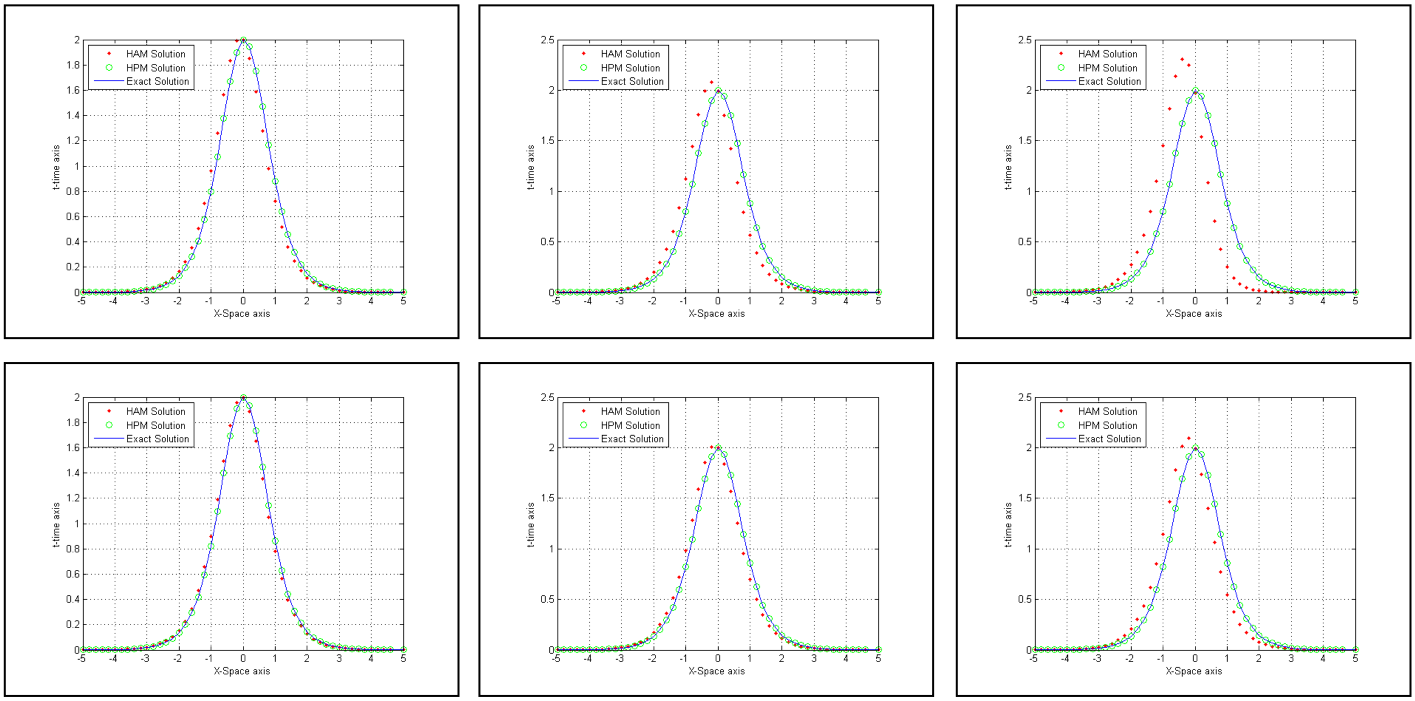
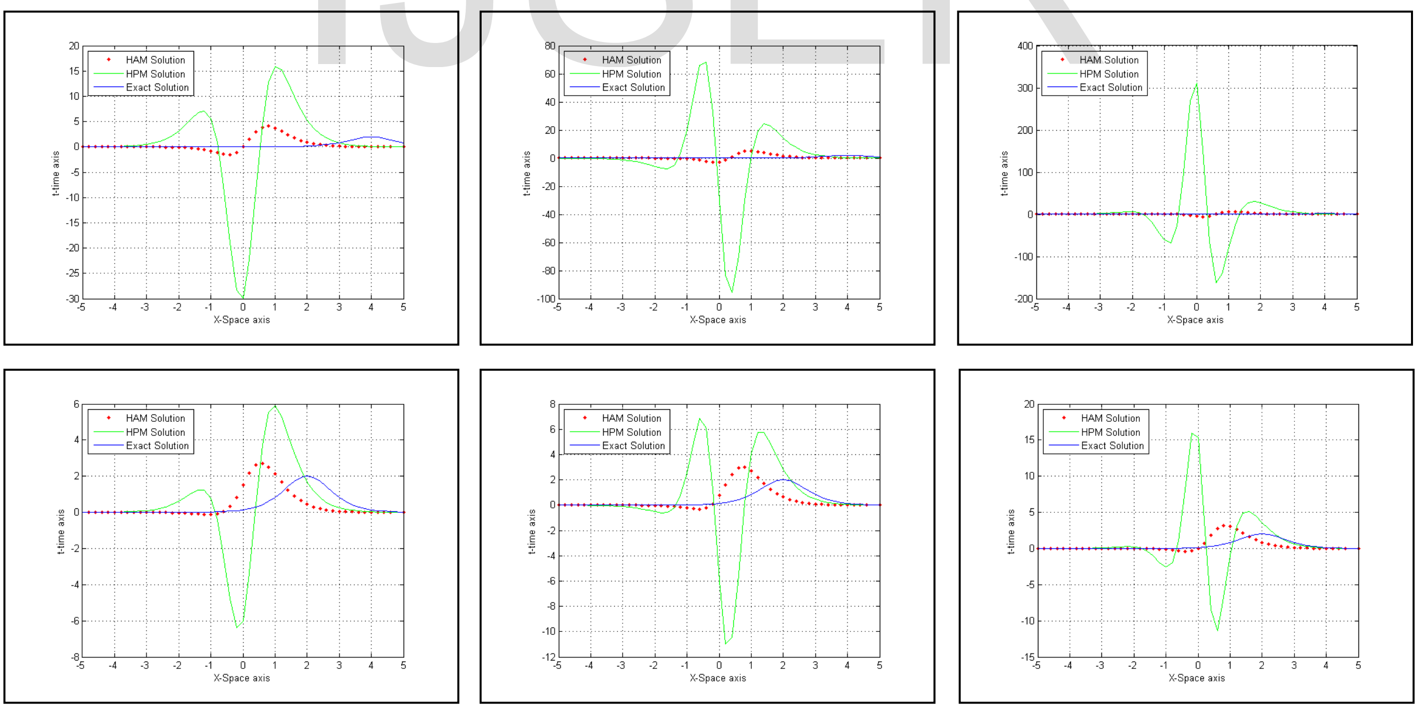
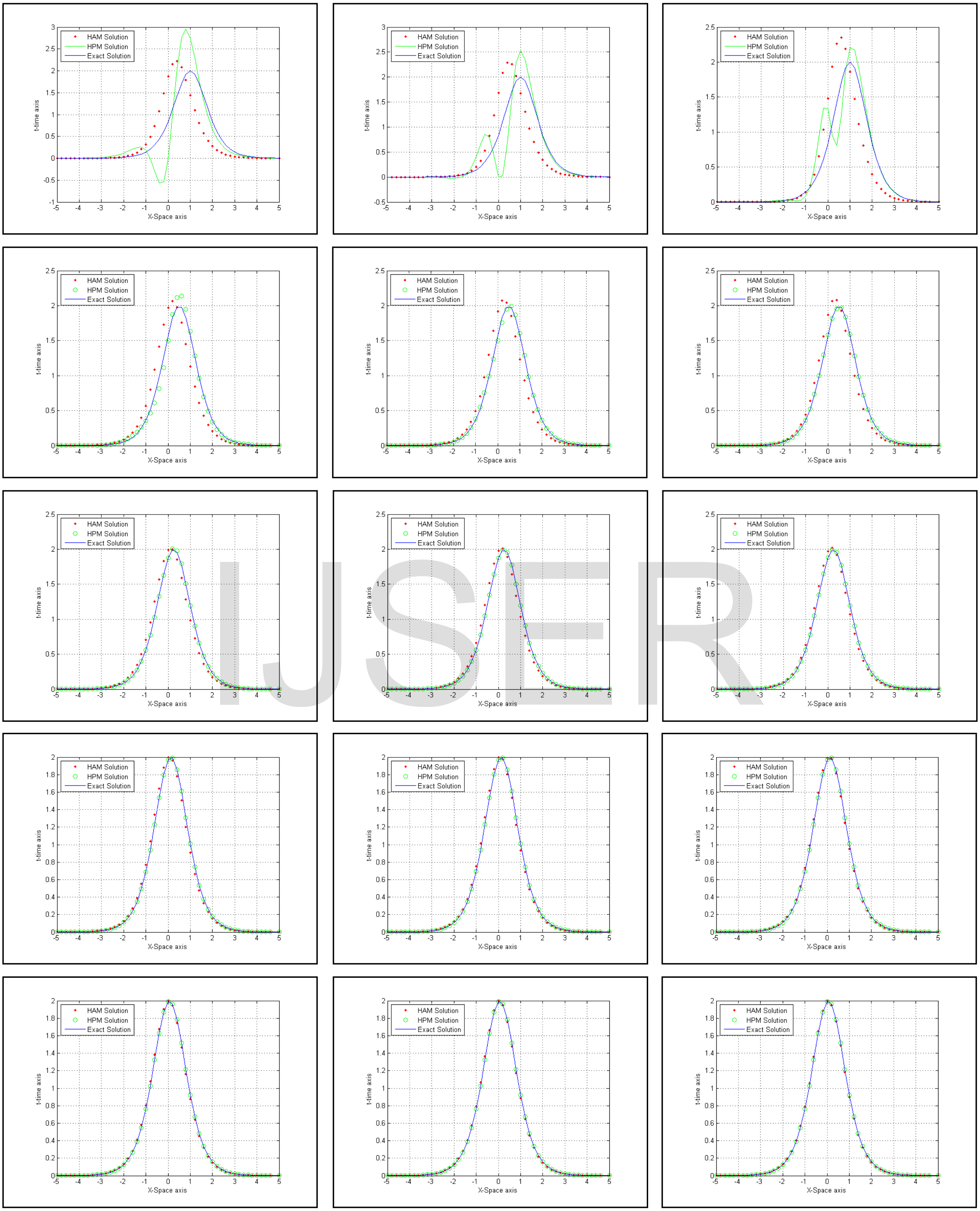


Fig. 4: 1st, 2nd and 3rd columns represent the exact solution, HAM solution and HPM solution of the K-dV equation for Case-I, Case-II and Case-III, respectively with regard to the values $(\mathcal{H}, \hbar) = (-1, -1), (1, 1), \left(-2, -\frac{1}{2}\right), \left(\frac{1}{2}, 2\right), \left(3, \frac{1}{3}\right)$, and $\left(-\frac{1}{3}, -3\right)$ in HAM.




HAM Solution
HPM Solution
Exact Solution
t-time axis
X-Space axis

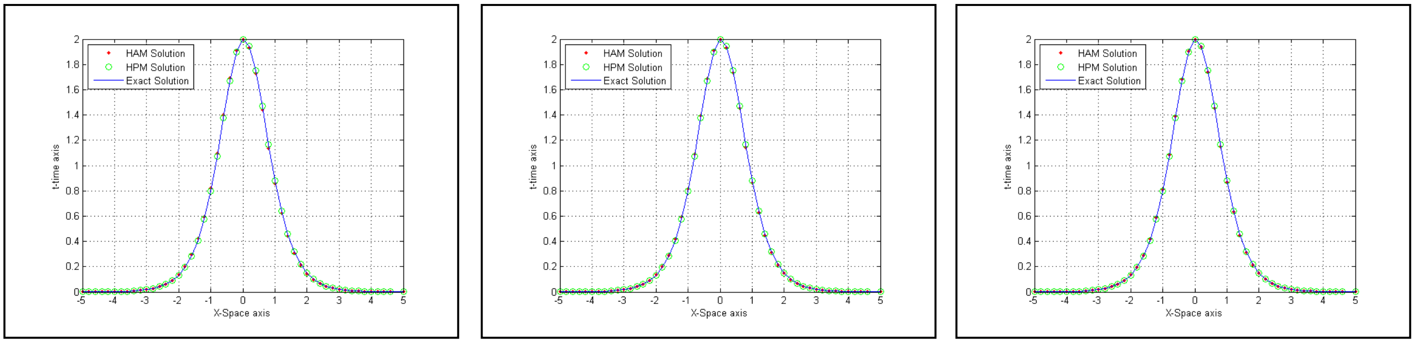
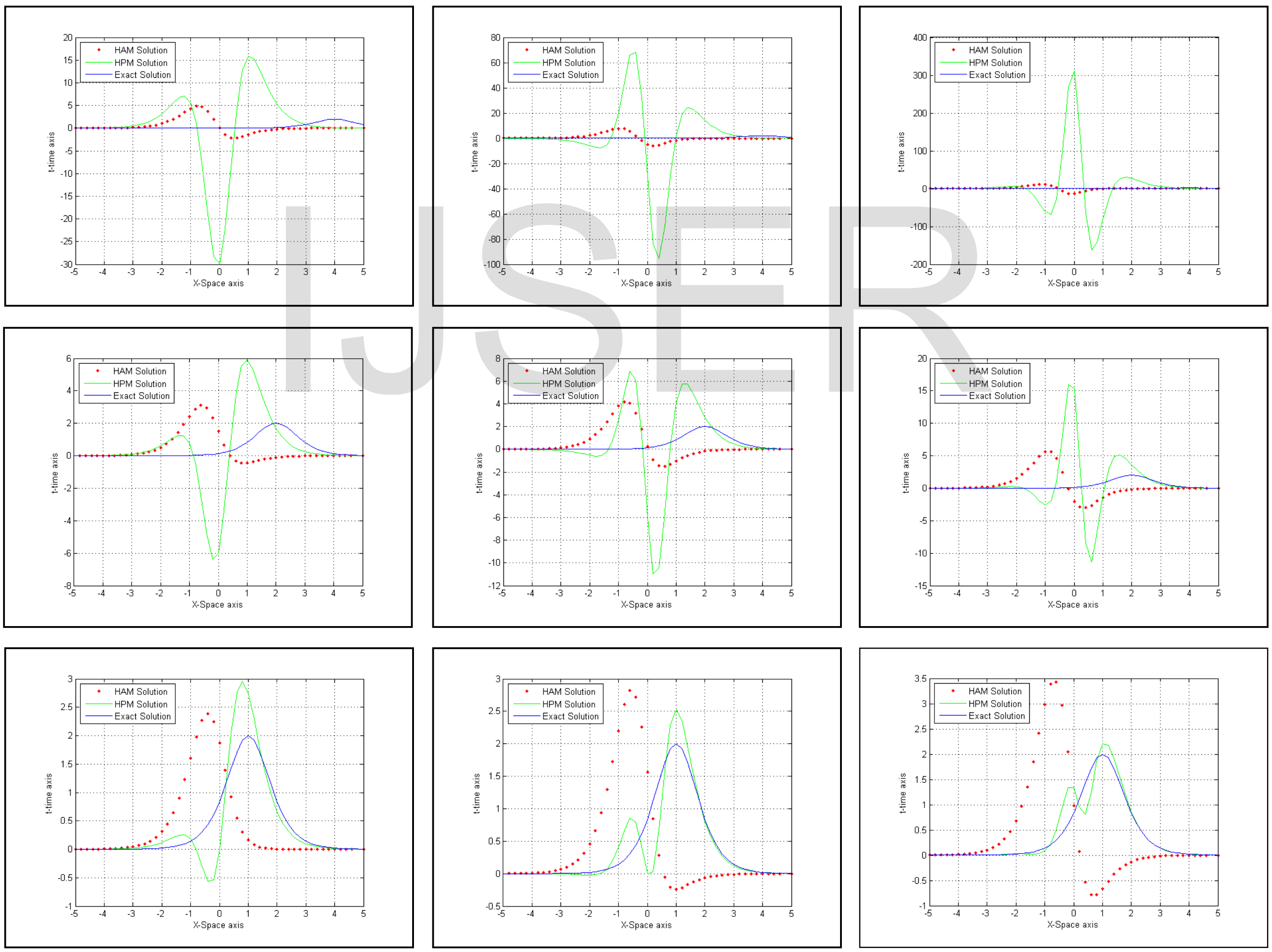


Fig. 5: 1st, 2nd and 3rd columns represent the exact solution, HAM solution and HPM solution of the K-dV equation for Case-I, Case-II and Case-III, respectively with regard to the values $(\mathcal{H}, \hbar) = (-0.5, 0.5), (0.5, -0.5), (-1, 0.25)$, and $(-0.25, 1)$ in HAM.



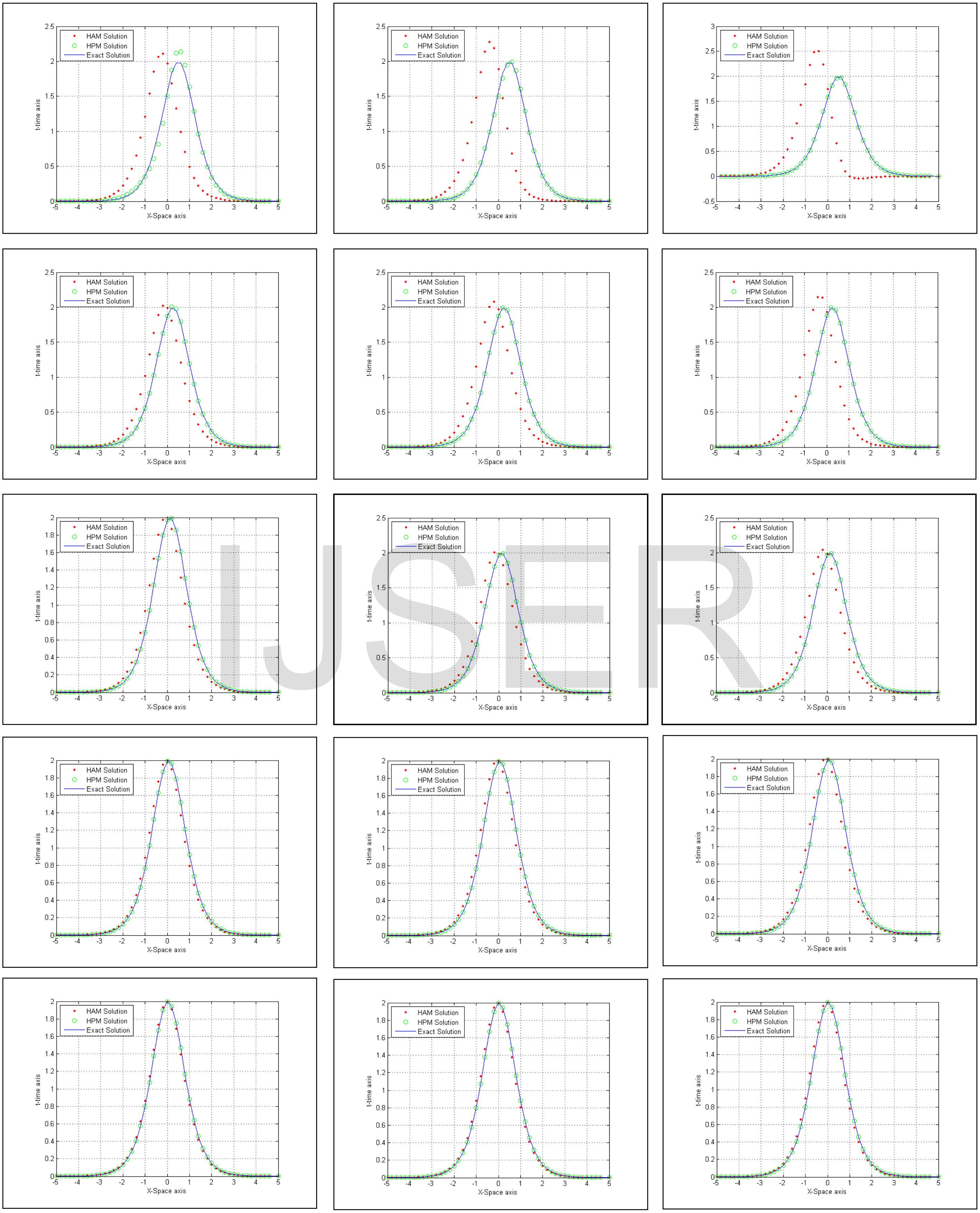


Fig. 6: 1st, 2nd and 3rd columns represent the exact solution, HAM solution and HPM solution of the K-dV equation for Case-I, Case-II and Case-III, respectively with regard to the values $(\mathcal{H}, \hbar) = (-0.5, -0.5), (0.5, 0.5), (1, 0.25)$, and $(-0.25, -1)$ in HAM.

TABLE 1

The l_2-error for HAM and HPM solution for all the three cases for the values$(\mathcal{H},\hbar) = (-1\,,1), (1,-1), \left(-2,\frac{1}{2}\right)$, $\left(-\frac{1}{2},2\right)$, $\left(3,-\frac{1}{3}\right)$, and $\left(\frac{1}{3},-3\right)$ considered in HAM.

Time t	Method	Case-I	Case-II	Case-III
2^0	HAM	62.718199989	193.753896419	577.584811761
	HPM	62.718199989	193.753896419	577.584811761
2^{-1}	HAM	14.841161323	22.282477407	32.612810041
	HPM	14.841161323	22.282477407	32.612810041
2^{-2}	HAM	2.631481915	1.973099259	1.438693389
	HPM	2.631481915	1.973099259	1.438693389
2^{-3}	HAM	0.369167277	0.138889796	0.050721759
	HPM	0.369167277	0.138889796	0.050721759
2^{-4}	HAM	0.047605331	0.008966728	0.001638531
	HPM	0.047605331	0.008966728	0.001638531
2^{-5}	HAM	0.005998090	0.000565086	0.000005164
	HPM	0.005998090	0.000565086	0.000005164
2^{-6}	HAM	0.000751258	0.000035392	0.000000005
	HPM	0.000751258	0.000035392	0.000000005

TABLE 2

The l_2-error for HAM and HPM solution for all the three cases for the values $(\mathcal{H},\hbar) = (-1\,,0.5), (1,-0.5)$, $(0.5,-1)$, and $(-0.5,1)$ considered in HAM.

Time t	Method	Case-I	Case-II	Case-III
2^0	HAM	19.862446963	33.769828542	70.544316243
	HPM	62.718199989	193.753896419	577.584811761
2^{-1}	HAM	7.061387460	7.276843876	8.137547253
	HPM	14.841161323	22.282477407	32.612810041
2^{-2}	HAM	2.200176138	1.621499716	1.058254270
	HPM	2.631481915	1.973099259	1.438693389
2^{-3}	HAM	0.744438007	0.441997160	0.253230788
	HPM	0.369167277	0.138889796	0.050721759
2^{-4}	HAM	0.311492307	0.165048927	0.089137213
	HPM	0.047605331	0.008966728	0.001638531
2^{-5}	HAM	0.147264288	0.074827306	0.038415451
	HPM	0.005998090	0.000565086	0.000005164
2^{-6}	HAM	0.072537104	0.036419152	0.018341876
	HPM	0.000751258	0.000035392	0.000000005
2^{-7}	HAM	0.036130509	0.018084119	0.009058835
	HPM	0.000093954	0.000002213	0.000000000

TABLE 3

The l_2-error for HAM and HPM solution for all the three cases for the values $(\mathcal{H},\hbar) = (-1\,,-0.5), (1,0.5), (0.5,1)$, and $(-0.5,-1)$ considered in HAM.

Time t	Method	Case-I	Case-II	Case-III
2^0	HAM	27.284422088	67.810320782	196.506643920
	HPM	62.718199989	193.753896419	577.584811761
2^{-1}	HAM	14.163199159	25.303782693	46.794994445
	HPM	14.841161323	22.282577407	32.612810041
2^{-2}	HAM	8.954242418	13.791224761	21.410765669
	HPM	2.631481915	1.973099259	1.438693389
2^{-3}	HAM	4.984952844	7.508981031	11.326092940
	HPM	0.369167277	0.138889796	0.050721759
2^{-4}	HAM	2.570545445	3.859229076	5.795564534
	HPM	0.047605331	0.008966728	0.001638531
2^{-5}	HAM	1.295560069	1.943748812	2.916426582
	HPM	0.005998090	0.000565086	0.000005164
2^{-6}	HAM	0.649083119	0.973675208	1.460612067
	HPM	0.000751258	0.000035392	0.000000162
2^{-7}	HAM	0.324704987	0.487063779	0.730608039
	HPM	0.000093954	0.000002213	0.000000005
2^{-8}	HAM	0.162372939	0.243560195	0.365341838
	HPM	0.000011746	0.000000138	0.000000000
2^{-9}	HAM	0.081189026	0.121783637	0.1826756483
	HPM	0.000001468	0.000000009	0.0000000000

TABLE 4

The l_2-error for HAM and HPM solution for all the three cases for the values $(\mathcal{H},\hbar) = (-1\,,-1), (1,1), \left(-2,-\frac{1}{2}\right), \left(\frac{1}{2},\ 2\right)$, $\left(3,\ \frac{1}{3}\right)$, and $\left(-\frac{1}{3},-3\right)$ considered in HAM.

Time t	Method	Case-I	Case-II	Case-III
2^0	HAM	81.739945215	339.130355542	1839.752583995
	HPM	62.718199989	193.753896419	577.584811761
2^{-1}	HAM	32.151611344	91.105747660	288.842157676
	HPM	14.841161323	22.282477407	32.612810041
2^{-2}	HAM	16.899705653	37.326398737	83.363702134
	HPM	2.631481915	1.973099259	1.438693389
2^{-3}	HAM	8.984843690	18.393021229	37.582020926
	HPM	0.369167277	0.1388897959	0.050721759
2^{-4}	HAM	4.585151957	9.222230737	18.532962159
	HPM	0.047605331	0.008966728	0.001638531
2^{-5}	HAM	2.305127771	4.616711291	9.244127321
	HPM	0.005998090	0.000565086	0.000005164
2^{-6}	HAM	1.154164220	2.309134279	4.619595362
	HPM	0.000751258	0.000035392	0.000000162
2^{-7}	HAM	0.577283143	1.154666980	2.309499457
	HPM	0.000093954	0.000002213	0.000000005
2^{-8}	HAM	0.288666731	0.577346049	1.154712774
	HPM	0.000011757	0.000000138	0.000000000

TABLE 5

The l_2-error for HAM and HPM solution for all the three cases for the values $(\mathcal{H}, \hbar) = (-0.5, 0.5), (0.5, -0.5), (-1, 0.25)$, and $(-0.25, 1)$ considered in HAM.

Time t	Method	Case-I	Case-II	Case-III
2^0	HAM	10.354100581	12.811010909	16.311437360
	HPM	62.718199990	193.753896420	577.584811761
2^{-1}	HAM	5.996894130	5.895856476	5.614762687
	HPM	14.841161323	22.282477407	32.612810041
2^{-2}	HAM	2.857547504	2.412555823	2.032031850
	HPM	2.631481915	1.973099259	1.438693389
2^{-3}	HAM	1.341834764	1.045717677	0.819790923
	HPM	0.369167277	0.138889796	0.050721759
2^{-4}	HAM	0.655278965	0.496609591	0.377323721
	HPM	0.047605331	0.008966728	0.001638531
2^{-5}	HAM	0.325493193	0.244771618	0.184203194
	HPM	0.005998090	0.000565086	0.000005164
2^{-6}	HAM	0.16247190	0.121935644	0.091530325
	HPM	0.000751258	0.000035392	0.000000162
2^{-7}	HAM	0.081201409	0.060911280	0.045693301
	HPM	0.000093954	0.000002213	0.000000005

TABLE 6

The l_2-error for HAM and HPM solution for all the three cases for the values $(\mathcal{H}, \hbar) = (-0.5, -0.5), (0.5, 0.5), (1, 0.25)$, and $(-0.25, -1)$ considered in HAM.

Time t	Method	Case-I	Case-II	Case-III
2^0	HAM	12.416474038	20.157536021	33.814347982
	HPM	62.718199989	193.753896419	577.584811761
2^{-1}	HAM	8.906696778	11.425235031	15.069595136
	HPM	14.841161323	22.282477407	32.612810041
2^{-2}	HAM	6.150968963	7.627559517	9.519356544
	HPM	2.631481915	1.973099259	1.438693389
2^{-3}	HAM	3.456631905	4.307124161	5.373854083
	HPM	0.369167277	0.138889796	0.050721759
2^{-4}	HAM	1.784580840	2.228791702	2.784413109
	HPM	0.047605331	0.008966728	0.001638531
2^{-5}	HAM	0.899633476	1.124292643	1.405158129
	HPM	0.005998090	0.000565086	0.000005164
2^{-6}	HAM	0.450744665	0.563399447	0.704223014
	HPM	0.000751258	0.000035392	0.000000162
2^{-7}	HAM	0.225488642	0.281856870	0.352317790
	HPM	0.000093954	0.000002213	0.000000005

6 CONCLUSION:

In this study, both HAM and HPM have been applied to solve the K-dV equation (a) and compared all the results obtained by these two methods with exact solution (b). From Fig. 1 and Table 1 we see that the HAM and HPM solution coincide when the values of $\mathcal{H}$ and $\hbar$ are taken in such a way that $\mathcal{H}\hbar = -1$. For the other setting the HPM gives better results than HAM because the initial approximation is chosen properly. We hope the study will be helpful for further studies of HAM and HPM for other differential equations.

ACKNOWLEDGEMENT:

I am very grateful to Md. Safik Ullah for his help and support in this work.

ETHICAL STATEMENT:

We have not intentionally engaged in or participate in any form of malicious harm to another person or animal. We have no conflict of interest.

REFERENCE:

[1] A. I. Enagi, M. Bawa and A. M. Sani, "Mathematical Study of Diabetes And Its Complication Using The Homotopy Perturbation Method". International Journal of Mathematics and Computer Science, Vol. 12, No. 1, pp. 43-63, 2017.

[2] A. Shirilord and M. dehghan, "The Use of Homotopy Analysis Method For Solving Generalized Sylvester Matrix Equation With Applications". Engineering With Computers, Vol. 38, pp. 2699-2716, 2022. doi: 10.1007/s00366-020-01219-0.

[3] A. A. Ayoade, O. J. Peter, A. I. Abioye, T. F. Aminu and O. A. Uwaheren, "Application of Homotopy Perturbation Method to An SIR Mumps_Model". Advances in Mathematics: Scientific Journal, Vol. 9, No. 3, pp. 1329-1340, 2020.

[4] A. Georgieva and I. Naydenova, "Application of Homotopy Analysis Method For Solving of Two Dimensional Linear Volterra Fuzzy Integral Equations". 2019. doi: 10.1063/1.5127477.

[5] A. K. Ray, B. Vasu, P. V. S. N Murthy and R. S. R. Gorla, "Non-Similar Solution of Eyring–Powell Fluid Flow And Heat Transfer With Convective Boundary Condition: Homotopy Analysis Method". Int. J. Appl. Comput. Math. 6, Vol. 16, 2020. doi: 10.1007/s40819-019-0765-1.

[6] A. Yokus, "Comparison of Caputo And Conformable Derivatives For Time-Fractional K-Dv Equation Via The Finite Difference Method". Int. J. of Modern Phys.B, Vol. 32, No. 29.

[7] R. Behrouz, "Application of He's Homotopy Perturbation Method And Variational Iteration Method For Nonlinear Partial Integro-Differential Equations". World Appl. Sc. J., Vol. 7, No. 4, pp. 399-404, 2009.

[8] B. Ren and J. Lin, "Soliton Molecules, Nonlocal Symmetry And CRE Method of The K-Dv Equation With Higher-Order Corrections". Phys. Scr., Vol. 95, 075202.

[9] D. D. Ganji, "The Application of He's Homotopy Perturbation Method to Nonlinear Equations Arising in Heat Transfer". Physics Letters A, Vol. 355, pp. 337-341, 2006.

[10] Hammack, L. Joseph, and H. Segur, "The Korteweg-de Vries Equation And Water Waves. Part 2. Comparison With Experiments". Journal of Fluid mechanics, Vol. 65, part 2, pp. 289-314, 1974.
[11] Hammack, L. Joseph, and H. Segur, "The Korteweg-de Vries Equation And Waterwaves. Part 3. Oscillatory Waves". Journal of Fluid Mechanics, Vol. 84, part 2, pp. 337-358, 1978.
[12] H. Zedan, W. Barakati, and N. Hamad, "The Application of The Homotopy Analysis Method And The Homotopy Perturbation Method to The Davey-Stewartson Equations And Comparison Between Them And Exact Solutions". Journal of applied mathematics, Vol. 2013, pp. 1-12, 2013.
[13] J. Biazar, H. Ghazvini, "Convergence of The Homotopy Perturbation Method For Partial Differential Equations". Nonlinear Analysis: Real world Applications, Vol. 10, pp. 2633-2640, 2009.
[14] J. Biazar, H. Ghazvini, "Exact Solutions For Nonlinear Schrödinger Equations By He's Homotopy Perturbation Method". Physics Letters A. Vol. 366, pp. 97-84, 2007.
[15] J. H. He, "Comparison of Homotopy Perturbation Method And Homotopy Analysis Method". appl. Math. Comput., Vol. 156, pp. 527-539, 2004.
[16] J. H. He, "The Homotopy Perturbation Method For The Solving Non-Linear Oscillators With Discontinuities". Appl. Math. Comp., Vol. 151, pp. 287-292, 2004.
[17] B. Jazbi and M .Moini, "Application of Homotopy Perturbation Method For Solving Schrödinger Equation". Iranian J. Math. Sc. Inf., Vol. 3, No.3, No.2, pp. 11-19, 2008.
[18] J. H. He., "Homotopy Perturbation Technique". Computer Methods in applied mechanics and engineering, Vol. 178, pp. 257-262, 1999.
[19] K. U. Tariq, M. Inc, A. Pashrashid, M. Zubair and L. Akinyemi, "On The Structure Of Unsteady K-Dv Model Arising In Shallow Water". Journal of Ocean Engineering and Science, 2022. doi: 10.1016/j.joes.2022.01.
[20] K. Brauer, "The Korteweg-De Vries Equation: History, Exact Solutions, And Graphical Representation". University of Osnabruck/Germany, 2000.
[21] Korteweg, D. Johannes and G. D. Vries, "Xli. On The Change of Form of Long Waves Advancing in A Rectangular Canal, And on A New Type of Long Stationary Waves". The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, Vol. 39, no. 240, pp. 422-443, 1895.
[22] M. S. H. Chowdhury, I. Hashem and O. Abdulaziz, "Comparison of Homotopy Analysis Method And Homotopy Perturbation Method For Purely Nonlinear Fin-Type Problems". Journal of communications in nonlinear science and numerical simulation, Vol. 14, pp. 371-378, 2009.
[23] M. Semary and H. Hassan, "The Homotopy Analysis Method for q-difference equations". Ain Shams Engineering Journal, pp. 1-7, 2016.
[24] N. H. Aljahdaly, A. Akgul, R Shah, I. Mahariq, and J. Kafle, "A Comparative Analysis of The Fractional-Order Coupled Korteweg–De Vries Equations With The Mittag–Leffler Law". Journal of Mathematics, Vol. 2022, 8876149.
[25] N. Najafi, "Homotopy Analysis Method to Solve Fuzzy Impulsive Fractional Differential Equations". Int. J. of Modern Mathematical Sciences, Vol. 18, pp.58-75, 2020.
[26] M. Nadeem, J. H. He and A. Islam , "The Homotopy Perturbation Method For Fractional Differential Equations: Part 1 Mohand Transform". International Journal of Numerical Methods for Heat & Fluid Flow, Vol. 31 No. 11, pp. 3490-3504, 2021.
[27] N. Anjum, J. H. He, "Two Modifications of The Homotopy Perturbation Method For Nonlinear Oscillators". J. Appl. Comput. Mech., Vol. 6, pp. 1420-1425, 2020.
[28] O. Zargar, M. M. Roozbahani, M. Bashirpour, and M. Baghani, "The Application of Homotopy Analysis Method to Determine The Thermal Response of Convective-Radiative Porous Fins With Temperature-Dependent Properties". Int. J. Appl. Mech., Vol. 11, No. 9, 1950089, 2019.
[29] P. Veeresha, D. G. Prakasha2, and J. Singh, "Solution For Fractional Forced K-Dv Equation Using Fractional Natural Decomposition Method". AIMS Mathematics, Vol. 5, pp. 798–810. doi:10.3934/math.2020054.
[30] P. A. Naik, J. Zu and M. Ghoreishi, "Stability Analysis And Approximate Solution of SIR Epidemic Model With Crowley-Martin Type Functional Response And Holling Type-Ⅱ Treatment Rate By Using Homotopy Analysis Method". Journal of Applied Analysis & Computation, Vol. 10, pp. 1482-1515, 2020. doi: 10.11948/20190239.
[31] P. Roul, "Application of Homotopy Perturbation Method to Biological Population Model". Application, Applied Math. Vol. 5, No. 2, pp. 272-281, 2010.
[32] S. A. El-Tantawy, A. H. Salas and W. Albalawi, "New localized and periodic solutions to a Korteweg–de Vries equation with power law nonlinearity: applications to some plasma models". Symmetry 2022, 14, 197.
[33] S. Liao, "Beyond Perturbation: Introduction to Homotopy Analysis Method". Chapman & Hall/CRC Press, Boca Raton, 2003.
[34] S. Liao, "The Proposed Homotopy Analysis Technique For The Solution of Nonlinear Problems". Ph. D. Thesis, Shanghai Jiao Tong University, 1992.
[35] S. Yildirim, "Homotopy Analysis Method For The Equation of The Type $\nabla^2 U = B(X,Y)$, And $\nabla^2 U = B(X,Y,U)$". Journal of applied mathematics and physics, Vol. 3, pp. 391-398, 2015.
[36] S. Liao, "Comparison Between The Homotopy Analysis Method And Homotopy Perturbation Method". Journal of applied mathematics and computation, Vol. 169, pp. 1186-1194, 2005.
[37] U. Biswal, S. Chakravrty and B. K. Ojha, "Application of Homotopy Perturbation Method in Inverse Analysis of Jeffery-Hamel Flow Problem". European Journal of Mechanics-B/Fluids, Vol. 86, pp. 107-112, 2021.
[38] Y. L. Sun, W. X. Ma and J. P. Yu, "N-Soliton Solutions And Dynamic Property Analysis of A Generalized Three-Component Hirota-Satsuma Coupled Kdv Equation". Appl. Mech. Lett., Vol. 120, 107224, 2021.